

Global Asymptotic Behavior of Positive Solutions on the System of Rational Difference Equations

$$x_{n+1} = 1 + x_n/y_{n-m}, \quad y_{n+1} = 1 + y_n/x_{n-m}$$

E. CAMOUZIS

Department of Mathematics

The American College of Greece, Deree College
6 Grivas Street, Aghia Paraskevi, 15342 Athens, Greece

G. PAPASCHINOPOULOS

Department of Electrical and Computer Engineering

Democritus University of Thrace

67100, Xanthi, Greece

(Received and accepted December 2002)

Communicated by R. P. Agarwal

Abstract—In this paper, we study the boundedness, persistence, and the global asymptotic behavior of the positive solutions of the system of two difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \dots,$$

where $x_i, y_i, i = -m, -m+1, \dots, 0$ are positive numbers and m is a positive integer. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Difference equations, Boundedness, Persistence, Global asymptotic behavior.

1. INTRODUCTION

In this paper, we study the global behavior of solutions of the system of difference equations

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \dots, \quad (1.1)$$

with initial conditions $x_i, y_i > 0, i = -m, -m+1, \dots, 0$ and m is a positive integer.

A pair of sequences of positive real numbers $\{(x_n, y_n)\}_{n=-m}^{\infty}$ that satisfies (1.1) is a positive solution of (1.1). If a positive solution of (1.1) is a pair of positive constants (x, y) , that solution is the equilibrium solution.

A positive solution $\{(x_n, y_n)\}_{n=-m}^{\infty}$ of (1.1) is bounded and persists if there exist positive constants M, N such that

$$M \leq x_n, y_n \leq N, \quad n = -m, -m+1, \dots$$

System (1.1) is part of a more general system of difference equations of the form

$$x_{n+1} = A + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \dots, \quad (1.2)$$

where A and B are positive constants.

The behavior of positive solutions of (1.2) was studied in [1] and the following result was exhibited.

THEOREM A. (See Lemma 2.2 of [1].) *Consider the system of difference equations (1.2). Then, the following statements are true.*

- (a) *If $A > 1$ and $B > 1$, every positive solution of (1.2) converges to the unique positive equilibrium solution of (1.2).*
- (b) *If $A < 1$ or $B < 1$, (1.2) possesses unbounded solutions.*

We can choose the initial conditions of (1.2) so that

$$x_n = y_n, \quad n = -m, -m+1, \dots, 0. \quad (1.3)$$

Then, $x_n = y_n$ for all $n \geq -m$ and so the system of difference equations (1.2) reduces into the rational recursive difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-m}}. \quad (1.4)$$

Equation (1.4) was studied in [2] and the following result was exhibited.

THEOREM B. (See [2].) *Let $A > 1$. Then, the unique positive equilibrium of equation (1.4) is globally asymptotically stable.*

When $m = 1$, equation (1.4) was studied in [3].

In the case where $A \leq 1$ and $m > 3$, the behavior of solutions of (1.4) has not been investigated so far. On the other hand, if we choose the initial conditions of (1.1) as in (1.3), the results of this manuscript imply that in the case where $A = 1$ and $m > 3$, every positive solution of (1.4) converges to the unique positive equilibrium.

First, we study the boundedness and persistence of the positive solutions of (1.1). Furthermore, we prove that (1.1) has an infinite number of positive equilibrium solutions and that every positive solution of (1.1) converges to a positive equilibrium solution of (1.1) as $n \rightarrow \infty$.

2. MAIN RESULTS

In the following lemma we show boundedness and persistence of the positive solutions of (1.1).

LEMMA 2.1. *Every positive solution of (1.1) is bounded and persists.*

PROOF. Let $\{(x_n, y_n)\}_{n=-m}^{\infty}$ be a positive solution of (1.1). Since $x_n > 0$ and $y_n > 0$ for all $n \geq -m$, (1.1) implies that

$$x_n > 1, \quad y_n > 1, \quad n = 1, 2, \dots \quad (2.1)$$

In view of (2.1), there exists a positive real number $L > 1$ such that

$$x_i, y_i \in \left[L, \frac{L}{L-1} \right], \quad i = 1, 2, \dots, m+1. \quad (2.2)$$

Then, from (1.1) and (2.2) we have

$$L = 1 + \frac{L}{L/(L-1)} \leq x_{m+2} = 1 + \frac{x_{m+1}}{y_1} \leq 1 + \frac{L/(L-1)}{L} = \frac{L}{L-1}$$

and

$$L = 1 + \frac{L}{L/(L-1)} \leq y_{m+2} = 1 + \frac{y_{m+1}}{x_1} \leq 1 + \frac{L/(L-1)}{L} = \frac{L}{L-1}.$$

Inductively, we get

$$x_i, y_i \in \left[L, \frac{L}{L-1} \right], \quad i = 1, 2, \dots \quad (2.3)$$

The proof is complete.

We now state our main theorem.

THEOREM 2.1. *Consider the system of difference equations (1.1). Then, the following statements are true.*

- (i) *Equation (1.1) has an infinite number of positive equilibrium solutions (x, y) with $x, y \in (1, \infty)$ that satisfy the equation*

$$xy = x + y. \quad (2.4)$$

- (ii) *Every positive solution of (1.1) converges to a positive equilibrium solution of (1.1) as $n \rightarrow \infty$.*

PROOF.

- (i) Let $x, y \in (1, \infty)$ such that (2.4) is satisfied. By rearranging (2.4) we have

$$x = 1 + \frac{x}{y}, \quad y = 1 + \frac{y}{x}$$

from which it follows that (x, y) is a positive equilibrium solution of (1.1). This completes the proof of (i).

- (ii) From (2.1) and (2.2), we have

$$\begin{aligned} l_1 &= \liminf_{n \rightarrow \infty} x_n \geq L > 1, & l_2 &= \liminf_{n \rightarrow \infty} y_n \geq L > 1, \\ L_1 &= \limsup_{n \rightarrow \infty} x_n > 1, & L_2 &= \limsup_{n \rightarrow \infty} y_n > 1. \end{aligned} \quad (2.5)$$

From (1.1) and (2.5), we have

$$l_1 \geq 1 + \frac{l_1}{L_2}, \quad l_2 \geq 1 + \frac{l_2}{L_1}, \quad L_1 \leq 1 + \frac{L_1}{l_2}, \quad L_2 \leq 1 + \frac{L_2}{l_1}$$

from which it follows that

$$l_1 = \frac{L_2}{L_2 - 1}, \quad l_2 = \frac{L_1}{L_1 - 1}. \quad (2.6)$$

In view of (2.5) there exists a sequence $n_s \geq 2m + 1$ such that

$$\lim_{s \rightarrow \infty} x_{n_s} = L_1, \quad \lim_{s \rightarrow \infty} x_{n_s+r} = a_r \leq L_1, \quad r = -2m-1, -2m, \dots, -1. \quad (2.7)$$

Using (1.1), we have

$$y_{n_s-m-1} = \frac{x_{n_s-1}}{x_{n_s}-1}. \quad (2.8)$$

By taking limits in (2.8) as $s \rightarrow \infty$ we get

$$\lim_{s \rightarrow \infty} y_{n_s-m-1} = b_2, \quad (2.9)$$

where b_2 is a real number greater than one. In view of (2.5), we have

$$b_2 \geq l_2. \quad (2.10)$$

In addition, with the use of (1.1), we have

$$x_{n_s} = 1 + \frac{x_{n_s-1}}{y_{n_s-m-1}}. \quad (2.11)$$

By taking limits in (2.11) as $s \rightarrow \infty$, and in view of (2.6)–(2.10), we have

$$L_1 = 1 + \frac{a_{-1}}{b_2} \leq 1 + \frac{L_1}{l_2} = L_1$$

from which it follows that

$$1 \geq \frac{a_{-1}}{L_1} = \frac{b_2}{l_2} \geq 1$$

and so

$$a_{-1} = L_1, \quad b_2 = l_2. \quad (2.12)$$

Therefore,

$$\lim_{s \rightarrow \infty} x_{n_s-1} = L_1, \quad \lim_{s \rightarrow \infty} y_{n_s-m-1} = l_2. \quad (2.13)$$

Inductively, it follows that

$$\begin{aligned} \lim_{s \rightarrow \infty} x_{n_s+i} &= L_1, & i &= -2m-1, -2m, \dots, 0, \\ \lim_{s \rightarrow \infty} y_{n_s+j} &= l_2, & j &= -3m-1, -3m, \dots, -m-1. \end{aligned} \quad (2.14)$$

At this point, we claim that $L_1 = l_1$. Suppose for the sake of contradiction that

$$L_1 > l_1.$$

Then, there exists $\epsilon > 0$ such that

$$L_1 - \epsilon > l_1. \quad (2.15)$$

Let

$$\epsilon_1 = \frac{\epsilon(l_2-1)}{L_1-\epsilon-1} > 0. \quad (2.16)$$

In view of (2.14), there exists $n_{s(\epsilon)} = n_{s_0} \in \{2m+1, 2m+2, \dots\}$ such that

$$\begin{aligned} L_1 - \epsilon &< x_{n_{s_0}+i} < L_1 + \epsilon, & i &= -2m-1, -2m, \dots, 0, \\ l_2 - \epsilon_1 &< y_{n_{s_0}-m-1} < l_2 + \epsilon_1. \end{aligned} \quad (2.17)$$

Furthermore, with the use of (1.1), we have

$$x_{n_{s_0}+1} = 1 + \frac{x_{n_{s_0}}}{y_{n_{s_0}-m}}, \quad y_{n_{s_0}-m} = 1 + \frac{y_{n_{s_0}-m-1}}{x_{n_{s_0}-2m-1}}. \quad (2.18)$$

In view of (2.6) and (2.16)–(2.18), we have

$$y_{n_{s_0}-m} < 1 + \frac{l_2 + \epsilon_1}{L_1 - \epsilon} = l_2 + \epsilon_1. \quad (2.19)$$

Also (2.6) and (2.16)–(2.19) imply

$$x_{n_{s_0}+1} > 1 + \frac{L_1 - \epsilon}{l_2 + \epsilon_1} = L_1 - \epsilon. \quad (2.20)$$

Inductively, it follows

$$y_{n_{s_0}+i} < 1 + \frac{l_2 + \epsilon_1}{L_1 - \epsilon} = l_2 + \epsilon_1, \quad i = -m, -m+1, \dots \quad (2.21)$$

and

$$x_{n_{s_0}+i} > 1 + \frac{L_1 - \epsilon}{l_2 + \epsilon_1} = L_1 - \epsilon, \quad i = 1, 2, \dots \quad (2.22)$$

By taking limits in (2.22), as $i \rightarrow \infty$, we get

$$l_1 \geq L_1 - \epsilon.$$

This contradicts (2.15) and so

$$l_1 = L_1. \quad (2.23)$$

In addition, from (2.6) and (2.23), we get

$$l_2 = L_2. \quad (2.24)$$

Therefore, the sequences $\{x_n\}_{n=-m}^\infty$ and $\{y_n\}_{n=-m}^\infty$ converge to finite limits x and y , respectively. In fact $x, y \in (1, \infty)$ and

$$\lim_{n \rightarrow \infty} x_n = x, \quad \lim_{n \rightarrow \infty} y_n = y. \quad (2.25)$$

By taking limits in (1.1) as $n \rightarrow \infty$ we have that (x, y) satisfies (2.4). Therefore, (x, y) is a positive equilibrium solution of (1.1). The proof of the theorem is complete.

We conclude this manuscript by posing the following open problem.

OPEN PROBLEM. Consider the system of difference equations (1.2) with $A < 1$ and $B < 1$. Find the set of all initial conditions that generates bounded solutions. In addition, investigate the global behavior of these solutions.

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