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Open Problems and Conjectures

Edited by Gerry Ladas

In this section we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas: gladas@math.uri.edu.

I pledge to donate the amount of $600 (USA) to the International Society of Difference Equations, provided that the complete solutions of the open problems and conjectures in this paper are brought to the attention of myself and the President of the Society by the end of the year 2005.

On third-order rational difference equations, part 5

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This paper is the fifth part in a series of manuscripts on 'Open Problems and Conjectures' dealing with third-order rational difference equations of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots \tag{1}$$

with nonnegative parameters and nonnegative initial conditions. See [1]–[3] and [6] for the preceeding parts.

In [6] we presented some powerful period-two trichotomy results which hold for some special cases of equation (1) and in [3] we listed all special cases of equation (1) which possess an essentially "unique" prime period-two solution.

Our main goal here is to present several open problems and conjectures on period-three solutions and period-three trichotomies of equation (1).

Open Problem 1 *Determine all special cases of equation* (1) *which possess prime period-three solutions and in each case determine the local and the global stability character of the period-three solutions.*

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A special case of equation (1) with period-three solutions and period-three trichotomies is the equation

$$x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots \tag{2}$$

with nonnegative parameters A, B, C, D and nonnegative initial conditions such that the denominator is always positive. The case $D = 0$ is contained in [4].

In the numbering system which was introduced in [6], equation (2) contains 15 equations. Five of them are trivial, namely,

#13, #14, #15, #16, and #37

and the remaining cases we list below in normalized form and with all parameters in the equation being positive:

$$\#35 \quad x_{n+1} = \frac{x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots$$

$$\#36 \quad x_{n+1} = \frac{x_{n-2}}{A + x_{n-1}}, \quad n = 0, 1, \ldots$$

$$\#38 \quad x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-1}}, \quad n = 0, 1, \ldots$$

$$\#39 \quad x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-2}}, \quad n = 0, 1, \ldots$$

$$\#40 \quad x_{n+1} = \frac{x_{n-2}}{Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots$$

$$\#113 \quad x_{n+1} = \frac{x_{n-2}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, \ldots$$

$$\#114 \quad x_{n+1} = \frac{x_{n-2}}{A + Bx_n + x_{n-2}}, \quad n = 0, 1, \ldots$$

$$\#115 \quad x_{n+1} = \frac{x_{n-2}}{A + Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots$$

$$\#116 \quad x_{n+1} = \frac{x_{n-2}}{Bx_n + Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots$$

$$\#136 \quad x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots.$$

Now we present several results and open problems and conjectures on the above equations.

For equation (#35) the following period-three trichotomy result is easily established.

(a) When

$$A > 1,$$

every solution of equation (#35) converges to zero.

(b) When

$$A = 1,$$

every solution of equation (#35) converges to a period-three solution of the form

$$\ldots, 0, 0, \phi, 0, 0, \phi, \ldots$$

with $\phi \geq 0$.

(c) When

$$A < 1,$$

equation (#35) has unbounded solutions.

Open Problem 2

(a) Determine all positive initial conditions x_{-2}, x_{-1} and x_0 such that the solution of the equation

$$x_{n+1} = \frac{x_{n-2}}{1 + x_n}, \quad n = 0, 1, \ldots \tag{3}$$

converges to zero.

(b) Determine all positive initial conditions x_{-2}, x_{-1}, x_0 through which the solution of equation (3) converges to the prime period-three solution

$$\ldots, 0, 0, 1, 0, 0, 1, \ldots.$$

(c) Determine the limit of the solution of equation (3) with initial conditions

$$x_{-2} = x_{-1} = x_0 = 1.$$

An unbounded solution of equation (#35) when $A < 1$ is

$$0, 0, 1, 0, 0, \frac{1}{A}, 0, 0, \frac{1}{A^2}, \ldots.$$

Conjecture 1 *Assume $A < 1$. Show that equation (#35) has positive unbounded solutions and that every positive bounded solution converges to $(1 - A)/A$.*

Similar trichotomy results can be established for equations (#36), (#38) and (#113).

We unify them in the following theorem, which for $A > 0$ is established in [4].

Theorem 1 *Assume $A, B, C \in [0, \infty)$ with $B + C > 0$. Then the solutions of the equation*

$$x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{4}$$

have the following period-three trichotomy behaviour.

(a) When

$$A > 1,$$

every solution of equation (4) converges to zero.

(b) When

$$A = 1,$$

every solution of equation (4) converges to a period-three solution of the form

$$\ldots, 0, 0, \phi, 0, 0, \phi, \ldots,$$

with $\phi \geq 0$.

(c) When

$$0 \leq A < 1,$$

equation (4) has unbounded solutions.

Some unbounded solutions of equation (4), when $A = 0$, can be explicitly exhibited. For example, the solution of equation (#38) with

$$x_{-2} = 0, \quad x_{-1} = x > 0, \quad \text{and} \quad x_0 = y > 0$$

is given by

$$\left.\begin{array}{r} x_{3n+1} = 0 \\ x_{3n+2} = \frac{(Bx)^{F_n}}{By^{F_{n+1}}} \\ x_{3n+3} = \frac{y^{F_{n+2}}}{(Bx)^{F_{n+1}}} \end{array}\right\} \quad \text{for } n = 0, 1, \ldots$$

with the exponents F_n being the Fibonacci numbers with $F_0 = F_1 = 1$.

Next we assume that the parameter D in equation (2) is positive and we rewrite it in the normalized form

$$x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots \tag{5}$$

with nonnegative parameters A, B, C and nonnegative initial conditions x_{-2}, x_{-1}, x_0. To avoid the degenerate case of equation (#16), we will also assume that $A + B + C > 0$.

Equation (5) has a unique equilibrium point when

$$A \geq 1$$

and also when

$$A = 0.$$

When $A \geq 1$, zero is the only equilibrium point of equation (5) and, clearly, zero in this case is globally asymptotically stable.

When $A = 0$, equation (5) reduces to

$$x_{n+1} = \frac{x_{n-2}}{Bx_n + Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots \tag{6}$$

and $\bar{x} = (1/(B + C + 1))$ is the only equilibrium point of equation (6). One can show that the equilibrium point of equation (6) is locally asymptotically stable when

$$B < 1 + \sqrt{2} \quad \text{and} \quad C < \frac{1 - 2B + \sqrt{5 + 4B}}{2}. \tag{7}$$

When $A \in (0, 1)$, equation (5) has two equilibrium points, namely, the zero equilibrium point, which is a repeller, and the positive equilibrium point

$$\bar{x} = \frac{1 - A}{B + C + 1}.$$

One can show that the positive equilibrium point of equation (5) is locally asymptotically stable when

$$A \in [0,1), \quad B < \frac{2 - A + \sqrt{A^2+8}}{2} \quad \text{and} \quad C < \frac{1 - 2B + \sqrt{5 + 4A + 4B(1-A)}}{2}. \quad (8)$$

CONJECTURE 2 *Assume that equation* (8) *holds. Show that the positive equilibrium of equation* (5) *is a global attractor of all positive solutions.*

CONJECTURE 3 *Assume that* $B < 1 + \sqrt{2}$. *Show that every positive solution of equation* (#39) *converges to* $(1/(1+B))$.

CONJECTURE 4 *Assume that* $C < ((1+\sqrt{5})/2)$. *Show that every positive solution of equation* (#40) *converges to* $(1/(1+C))$.

Open Problem 3

(a) Determine the global character of solutions of equation (#39) when

$$B \geq 1 + \sqrt{2}.$$

(b) Determine the global character of solutions equation (#40) when

$$C \geq \frac{1+\sqrt{5}}{2}.$$

We have confirmed Conjectures 3 and 4 for

$$B, C \in [0,1).$$

Actually, the following result can be established.

THEOREM 2 *Assume that*

$$A, B + C \in [0,1).$$

Then every positive solution of equation (5) *converges to the positive equilibrium of equation* (5).

Proof. Let $\{x_n\}$ be a positive solution of equation (5). We will first establish that there exist positive numbers m, M and N sufficiently large such that

$$x_n \in [m, M] \quad \text{for} \;\; n \geq N. \quad (9)$$

To this end, when $A > 0$, note that

$$x_{n+1} \leq \frac{x_{n-2}}{A + x_{n-2}}, \quad n = 0, 1, \ldots$$

and so

$$\limsup_{n\to\infty} x_n \leq 1 - A.$$

Choose any positive number

$$\epsilon < \frac{(1-A)(1-B-C)}{B+C}$$

and set $M = 1 - A + \epsilon$. Next choose $N \geq 2$ and $m > 0$ such that

$$x_{N-2}, x_{N-1}, x_N \in (0, M)$$

and

$$m < \min\{x_{N-2}, x_{N-1}, x_N, (1-A)(1-B-C) - \epsilon(B+C)\}$$

Then

$$m < \frac{m}{A + (B+C)(1-A+\epsilon) + m} < x_{N+1} = \frac{x_{N-2}}{A + Bx_N + Cx_{N-1} + x_{N-2}}$$
$$< \frac{1-A+\epsilon}{A + (B+C)m + 1 - A + \epsilon} < 1 - A + \epsilon = M.$$

Therefore, by induction, equation (9) is satisfied.

On the other hand, when $A = 0$, choose $m > 0$ such that

$$x_{-2}, x_{-1}, x_0 \in [m, M]$$

where $M = 1 - (B+C)m$. Then

$$m \leq \frac{m}{(B+C)[1(B+C)m] + m} \leq x_1 = \frac{x_{-2}}{A + Bx_0 + Cx_{-1} + x_{-2}}$$
$$\leq \frac{1-(B+C)m}{(B+C)m + 1 - (B+C)m} \leq 1 - (B+C)m = M.$$

Therefore, by induction, equation (9) is again satisfied. Now it follows from equation (9) that the numbers

$$S = \limsup_{n\to\infty} x_n \quad \text{and} \quad I = \liminf_{n\to\infty} x_n$$

exist and they are both positive real numbers. Furthermore, from equation (5) it follows that

$$S \leq \frac{S}{A + (B+C)I + S}, \quad I \geq \frac{I}{A + (B+C)S + I}$$

and so $S = I$. The proof is complete. □

Conjecture 5 *Assume that*

$$B > 123.$$

Show that every positive solution of equation (#39) *converges to a periodic solution of period* 19.

CONJECTURE 6 *Assume that*

$$C > 8.$$

Show that every positive solution of equation (#40) *converges to a periodic solution of period* 13.

By using the identity

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

one can show that equation (5) has positive prime period-three solutions if and only if

$$0 \le A < 1 \quad \text{and} \quad B = C = 1. \tag{10}$$

All other possible prime period-three solutions of equation (5) are of the form

$$\ldots, 0, 0, \phi, 0, 0, \phi, \ldots \tag{11}$$

with $\phi \in (0, \infty)$ or of the form

$$\ldots, 0, \phi, \psi, 0, \phi, \psi, \ldots \tag{12}$$

with $\phi, \psi, \in (0, \infty)$.

One can see that equation (5) has prime period-three solutions of the form equation (11) if and only if

$$A \in (0, 1) \quad \text{or} \quad A = 0 \quad \text{and} \quad B, C \in (0, \infty). \tag{13}$$

Furthermore when equation (13) holds, equation (5) has a unique prime period-three solution of the form (11) with $\phi = 1 - A$.

Also equation (5) has prime period-three solutions of the form (12) if and only if

$$0 \le A < 1 \quad \text{and} \quad B = C = 1 \tag{14}$$

or

$$0 \le A < 1 \quad \text{and} \quad B, C \in (1, \infty) \tag{15}$$

or

$$0 \le A < 1 \quad \text{and} \quad B, C \in [0, 1). \tag{16}$$

Furthermore, when equation (14) holds, the values of ϕ, ψ in equation (12) are all positive numbers ϕ and ψ such that

$$\phi + \psi = 1 - A$$

and equation (5) has infinitely many period-three solutions.

When equation (15) or (16) holds, the values of ϕ, ψ in equation (12) are

$$\phi = \frac{(1-A)(1-C)}{1-BC} \quad \text{and} \quad \psi = \frac{(1-A)(1-B)}{1-BC}. \tag{17}$$

and equation (5) has a unique prime period-three solution in this case.

When equation (10) holds, the positive prime period-three solutions of equation (5)

$$\ldots, \phi, \psi, \omega, \ldots \tag{18}$$

are given by

$$\phi + \psi + \omega = 1 - A$$

with

$$\phi, \psi, \omega \in (0, 1) \quad \text{and} \quad (\phi, \psi, \omega) \neq \left(\frac{1-A}{3}, \frac{1-A}{3}, \frac{1-A}{3}\right).$$

In view of the above we see that when $A \in [0, 1)$, all prime period-three solutions of the equation

$$x_{n+1} = \frac{x_{n-2}}{A + x_n + x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots \tag{19}$$

are of the form (18) where ϕ, ψ, ω are all solutions of the equation

$$\phi + \psi + \omega = 1 - A$$

with

$$\phi, \psi, \omega \in [0, 1] \quad \text{and} \quad (\phi, \psi, \omega) \neq \left(\frac{1-A}{3}, \frac{1-A}{3}, \frac{1-A}{3}\right).$$

The following result shows that when

$$0 \leq A < 1$$

every solution of equation (19) converges to a (not necessarily prime) period-three solution of equation (19).

THEOREM 3 *Assume that* $A \in [0, 1)$. *Then every solution of equation* (19) *converges to a* (*not necessarily prime*) *period-three solution of equation* (19).

Proof. Note that

$$x_{n+1} - x_{n-2} = \frac{x_{n-2}}{A + x_n + x_{n-1} + x_{n-2}}(1 - A - x_{n-1} - x_{n-2}), \quad n = 0, 1, \ldots.$$

Set

$$J_n = 1 - A - x_n - x_{n-1} - x_{n-2}.$$

Then for $n \geq 0$,

$$J_{n+1} = 1 - A - x_n - x_{n-1} - \frac{x_{n-2}}{A + x_n + x_{n-1} + x_{n-2}} = \frac{A + x_n + x_{n-1}}{A + x_n + x_{n-1} + x_{n-2}} J_n.$$

Hence the signum of J_n is constant from which the result follows because every solution of equation (19) is bounded. □

Open Problem 4 *Determine the set of all positive initial conditions through which the solutions of equation* (19) *converge to a prime period-three solution.*

Open Problem 5 *Assume that equation* (13) *is satisfied. Determine the global character of solutions of equation* (5) *and, in particular, determine the basin of attraction of the period-three solution*

$$\ldots, 0, 0, 1 - A, \ldots.$$

Open Problem 6 *Assume that equation* (15) *or* (16) *is satisfied. Determine, in each case, the global character of solutions of equation* (5) *and in particular, determine the basin of attraction of each of the period-three solution*

$$\ldots, 0, 0, 1 - A, \ldots \quad \ldots, \frac{(1-A)(1-C)}{(1-BC)}, \frac{(1-A)(1-B)}{(1-BC)}, 0, \ldots.$$

It is not difficult to see that when equation (13) holds, the basin of attraction of the period-three solution

$$\ldots, 0, 0, 1 - A, \ldots.$$

includes all solutions of equation (5) with two of the three initial conditions x_{-2}, x_{-1}, x_0 equal to zero and the third positive.

We have also shown that when equation (15) or (16) holds, every solution of equation (5) with one of the three initial conditions x_{-2}, x_{-1}, x_0 equal to zero and the other two positive converges to a period-three solution of equation (5).

CONJECTURE 7 *Assume that*

$$0 \leq A < 1 \quad \textit{and} \quad B, C \in (1, \infty).$$

Then every positive solution of equation (5) converges to a period-three solution of equation (5).

Open Problem 7

(a) Investigate the behaviour of solutions of equation (#114) when $A < 1 \quad \textit{and} \quad B \geq 1$.
(b) Investigate the behaviour of solutions of equation (#115) when $A < 1 \quad \textit{and} \quad C \geq 1$.
(c) Investigate the behaviour of solutions of equation (#116) when either $B \neq 1$ or $C \neq 1$.

Open Problem 8 *Assume that k is a positive integer and* $A, B_0, \ldots, B_k \in [0, \infty)$. *Determine the global stability of the periodic solutions of the difference equation*

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + \ldots + B_k x_{n-k}}, \quad n = 0, 1, \ldots$$

and the global stability character of its equilibrium points.

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