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Open Problems and Conjectures

Gerry Ladas Editor

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Open Problems and Conjectures

Edited by Gerry Ladas

In this section we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas: gladas@math.uri.edu

I pledge to donate the amount of $600 (USA) to the International Society of Difference Equations, provided that the complete solutions of the open problems and conjectures in this paper are brought to the attention of myself and the President of the Society by the end of the year 2005.

On third-order rational difference equations, part 6

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This paper is the sixth part in a series of manuscripts on ‘OPEN PROBLEMS AND CONJECTURES’ dealing with third-order rational difference equations of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots \tag{1}$$

with nonnegative parameters and nonnegative initial conditions such that $\alpha + \beta + \gamma + \delta > 0$ and the denominator is always positive. See [4,9,11,20,29] for the preceeding parts.

In this paper, we address the boundedness character of each of the 225 special cases which are contained in equation (1) and in each case we either state a reference where the equation was investigated, establish our assertion, or offer a conjecture. The amazing thing is the simplicity and generality of the rules which characterize the boundedness behaviour of so many equations with so much diversity.

In the sequel we will use the notation which was introduced in [29], see also Appendix I(c) in [20]. In some cases, we will write the equation in normalized form with as few parameters as possible.

To start with, we conjecture that out of the 225 special cases which are contained in equation (1), 135 special cases involve equations with the property that every solution of the equation is bounded. For each of the remaining 90 special cases, there is some range of

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the parameters in which the equation has only bounded solutions, and in the complement of that range there exist both bounded and unbounded solutions.

The table in Appendix I summarizes the facts and our conjectures on the boundedness of each of the 225 special cases of equation (1). This appendix is based on a thorough analysis of the existing literature, on numerous computer observations and on many analytic investigations including the three new theorems, namely, Theorems 2, 3 and 4, which we present here. A glance at any special case will immediately reveal whether the equation has unbounded solutions in some range of its parameters. At a glance we will also know whether the boundedness character of the equation is an established result or still a conjecture.

1. Some straightforward cases

Clearly, all 14 special cases of equation (1) which are linear but nontrivial have unbounded solutions in some range of their parameters. They are the following equations:

#5, #9, #13, #41, #45, #49, #53,
#57, #61, #117, #121, #125, #129, #137.

See also the table in Appendix I.

Also, all 4 trivial and linear cases of equation (1) have only bounded solutions. They are the following equations:

#1, #6, #11, #16.

See also the table in Appendix I.

All 15 Riccati or Riccati-type special cases of equation (1) have only bounded solutions because their Riccati number is less than or equal to 1/4, or because every solution of the equation is periodic. See [28], p.17. They are the following equations:

#2, #3, #4, #17, #18, #19, #23, #30,
#37, #42, #47, #52, #65, #72, #79.

See also the table in Appendix I.

One can see that every special case of equation (1) for which all of the terms in the numerator are also contained in the denominator has only bounded solutions. By this we mean that if the constant α is present in the numerator of this special case, so is the constant A in the denominator. If the coefficient β of x_n is present in the numerator, so is the coefficient B of x_n in the denominator, and so on. This idea establishes the boundedness in the following 51 additional equations:

#26, #27, #32, #34, #39, #40, #86, #93,
#100, #101, #102, #103, #105, #106, #108, #109,
#111, #112, #114, #115, #116, #133, #134, #135,
#136, #141, #142, #145, #147, #150, #151, #156,
#158, #160, #163, #164, #189, #190, #191, #192,
#193, #194, #201, #206, #211, #216, #217, #218,
#219, #220, #225.

See also the table in Appendix I.

2. Second order rational difference equations

The global behaviour of solutions of the second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{2}$$

was investigated in [28]. Equation (2) contains 49 special cases. The boundedness character of 47 of these cases can be inferred from the work in [28]. The boundedness character of the remaining two cases, namely,

#166, #168

was determined in [25], where it was shown that every solution of each of these two equations is bounded. See also the table in Appendix I. The following theorem describes the boundedness character of solutions of all nontrivial special cases of equation (2).

THEOREM 1 (See [4,25,28])
(a) *Assume that*

$$C > 0.$$

Then every solution of equation (2) *is bounded.*
(b) *Assume that*

$$C = 0 \quad \textit{and} \;\; B > 0. \tag{3}$$

Then every solution of equation (2) *is bounded if and only if*

$$\gamma \leq \beta + A.$$

Equivalently, when equation (3) *holds, equation* (2) *has unbounded solutions if and only if*

$$\gamma > \beta + A.$$

3. Equations with bounded solutions only

The following theorem establishes the boundedness of all solutions in 12 new special cases, namely

#76, #81, #82, #144, #148, #152,
#175, #182, #204, #208, #212, #224.

See also the table in Appendix I. The proof is along the lines of Lemma 2 in [25] and will be omitted.

THEOREM 2 *The following statements are true.*

(a) *Assume that*

$$\alpha, \beta, \delta \in [0, \infty) \quad \textit{and} \quad B, D \in (0, \infty).$$

Then every solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \delta x_{n-2}}{Bx_n + Dx_{n-2}}, \quad n = 0, 1, \ldots$$

is bounded.

(b) *Assume that*

$$\alpha, \gamma, \delta \in [0, \infty) \quad \text{and} \quad C, D \in (0, \infty).$$

Then every solution of the equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots$$

is bounded.

(c) *Assume that*

$$\alpha, \beta, \gamma, \delta \in [0, \infty) \quad \text{and} \quad B, C, D \in (0, \infty).$$

Then every solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \ldots$$

is bounded.

Are there other special cases left with the property that every solution of the equation is bounded? We conjecture that the answer is "YES" and that they are the following 31 special cases:

#58,	#63,	#77,	#78,	#88,	#89,	#90,	#91,
#94,	#96,	#122,	#127,	#131,	#139,	#143,	#155,
#159,	#170,	#171,	#172,	#173,	#176,	#178,	#184,
#188,	#196,	#200,	#203,	#207,	#215,	#223.	

See also the table in Appendix I.

Open Problem 1 *For each of the* 31 *equations in the above list, determine the boundedness character of the equation and investigate the global stability of its equilibrium point*(*s*)*.*

Open Problem 2 *Extend and generalize Theorem* 2*.*

4. Equations with unbounded solutions

First we present a conjecture about a quite general equation which possesses unbounded solutions in some range of its parameters.

CONJECTURE 1 *Assume that*

$$\gamma, B + D \in (0, \infty) \quad \text{and} \quad \alpha, \beta, \delta, A, B, D \in [0, \infty). \tag{4}$$

Then the rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Dx_{n-2}}, \quad n = 0, 1, \ldots \tag{5}$$

has unbounded solutions in some range of its parameters and in particular when

$$\gamma > \beta + \delta + A. \tag{6}$$

This conjecture has already been confirmed in each of the following cases:

(i) $D = 0$; See [10,28] or Appendix II.
(ii) $\beta = B = 0$; See [12] or Appendix II.
(iii) $\beta = \delta = 0$; See [13] or Appendix II.
(iv) #87; See [17].
(v) #99; See [6].
(vi) #162; See [7].

The following theorem confirms Conjecture 1 when the parameters B and D are both positive. The proof, which is similar to the proof given in [10] for the case $D = 0$, but more tedious, will be given elsewhere.

Theorem 3 *Assume that*

$$\alpha, \beta, \delta, A \in [0, \infty) \quad \textit{and} \quad \gamma, B, D \in (0, \infty).$$

Then equation (5) *has unbounded solutions in some range of its parameters and in particular when equation* (6) *holds.*

Conjecture 1 pertains to the following 48 special cases of equation (1):

#10,	#12,	#29,	#31,	#33,	#46,	#48,	#54,
#56*,	#62,	#64,	#71,	#73,	#75,	#83,	#85*,
#87,	#95,	#97,	#99,	#110,	#118,	#120*,	#126,
#128,	#130,	#132*,	#138,	#140*,	#146,	#154,	#162,
#165,	#167*,	#169,	#177,	#179,	#181,	#183,	#185*,
#187,	#195,	#197*,	#199,	#202,	#210,	#214,	#222.

The conjecture has already been confirmed for the 40 equations without an asterisk. The eight cases with asterisks still remain to be confirmed or refuted. See also the table in Appendix I.

Open Problem 3 *For each of the eight equations with an asterisk in the above list, determine the region of parameters where every solution of the equation is bounded. Furthermore, investigate the boundedness character of solutions of equation* (5) *when*

$$\gamma \le \beta + \delta + A.$$

Open Problem 4 *Extend and generalize Theorem* 3.

The following result establishes the existence of unbounded solutions for equation (1) when $D = 0$ and δ, B, and C are positive. Without loss of generality we assume that $C = 1$.

THEOREM 4 *Assume that*

$$\delta, B \in (0, \infty) \quad \textit{and} \quad \alpha, \beta, \gamma, A \in [0, \infty). \tag{7}$$

Then the rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{8}$$

has unbounded solutions in some range of its parameters and in particular, when

$$\delta > A + \gamma B + \frac{\beta}{B}.$$

Proof Choose positive numbers m and ϵ such that

$$m \in \left(0, \delta - A - \gamma B - \frac{\beta}{B}\right) \quad \text{and} \quad \epsilon \in \left(0, \frac{m}{1+B}\right).$$

Set

$$K = \frac{1}{\epsilon}\left[\alpha + \beta\left(\epsilon + \frac{\beta}{B}\right) + \delta(\epsilon + \gamma)\right]$$

and

$$L = \frac{1}{\epsilon B}\left[\alpha + \gamma(\epsilon + \gamma) + \delta\left(\epsilon + \frac{\beta}{B}\right)\right].$$

Let $\{x_n\}$ be a solution of equation (8) with initial conditions chosen as follows:

$$x_{-2} > \max\{K, L\}, x_{-1} \in \left(0, \epsilon + \frac{\beta}{B}\right), \quad \text{and} \quad x_0 \in (0, \epsilon + \gamma).$$

Then we claim that

$$\lim_{n\to\infty} x_{3n+1} = \infty, \quad \lim_{n\to\infty} x_{3n+2} = \frac{\beta}{B}, \quad \text{and} \quad \lim_{n\to\infty} x_{3n+3} = \gamma.$$

Indeed,

$$\begin{aligned} x_1 &= \frac{\alpha + \beta x_0 + \gamma x_{-1} + \delta x_{-2}}{A + Bx_0 + x_{-1}} > \frac{\alpha + \beta x_0 + \gamma x_{-1}}{A + \gamma B + \frac{\beta}{B} + \epsilon(1+B)} + \frac{\delta x_{-2}}{A + \gamma B + \frac{\beta}{B} + \epsilon(1+B)} \\ &> \frac{\alpha + \beta x_0 + \gamma x_{-1}}{\delta} + \frac{\delta}{A + \gamma B + \frac{\beta}{B} + m} x_{-2}, \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\alpha + \beta x_1 + \gamma x_0 + \delta x_{-1}}{A + Bx_1 + x_0} < \frac{\alpha + \gamma(\varepsilon + \gamma) + \delta\left(\varepsilon + \frac{\beta}{B}\right) + \beta x_1}{Bx_1} \\ &< \frac{\alpha + \gamma(\epsilon + \gamma) + \delta\left(\epsilon + \frac{\beta}{B}\right) + \beta L}{BL} = \varepsilon + \frac{\beta}{B}, \end{aligned}$$

and

$$x_3 = \frac{\alpha + \beta x_2 + \gamma x_1 + \delta x_0}{A + Bx_2 + x_1} < \frac{\alpha + \beta\left(\epsilon + \frac{\beta}{B}\right) + \delta(\epsilon + \gamma) + \gamma x_1}{x_1}$$
$$< \frac{\alpha + \beta\left(\epsilon + \frac{\beta}{B}\right) + \delta(\epsilon + \gamma) + \gamma K}{K} = \epsilon + \gamma.$$

It follows by induction that for $n \geq 0$,

$$x_{3n+1} > \frac{\alpha + \beta x_{3n} + \gamma x_{3n-1}}{\delta} + \frac{\delta}{A + \gamma B + \frac{\beta}{B} + m} x_{3n-2},$$

$$x_{3n-2} < \epsilon + \frac{\beta}{B},$$

and

$$x_{3n+3} < \epsilon + \gamma.$$

Therefore

$$\lim_{n\to\infty} x_{3n+1} = \infty,$$

$$x_{3n+2} = \frac{\frac{\alpha}{x_{3n+1}} + \beta + \gamma\frac{x_{3n}}{x_{3n+1}} + \delta\frac{x_{3n-1}}{x_{3n+1}}}{\frac{A}{x_{3n+1}} + B + \frac{x_{3n}}{x_{3n+1}}} \to \frac{\beta}{B} \quad \text{as} \quad n \to \infty,$$

and

$$x_{3n+3} = \frac{\frac{\alpha}{x_{3n+1}} + \beta\frac{x_{3n+2}}{x_{3n+1}} + \gamma + \delta\frac{x_{3n}}{x_{3n+1}}}{\frac{A}{x_{3n+1}} + B\frac{x_{3n+2}}{x_{3n+1}} + 1} \to \gamma \quad \text{as} \quad n \to \infty$$

and the proof is complete. □

Theorem 4 establishes the existence of unbounded solutions for each of the following 16 equations:

#38, #80, #92, #98, #113, #149, #157, #161,
#174, #180, #186, #198, #205, #209, #213, #221.

For two of the above equations, namely #38 and #113, the existence of unbounded solutions is also a consequence of the period-3 trichotomy known for third-order equations. See Appendix III and [9]. See also the table in Appendix I.

Open Problem 5 *Assume that equation* (7) *holds. Determine the boundedness character of solutions of equation* (8) *when*

$$\delta \leq A + \gamma B + \frac{\beta}{B}$$

and determine the global stability of its equilibrium point(*s*).

Open Problem 6 *Extend and generalize Theorem* 4.

In addition to the period-two trichotomies which are known for some special cases of equation (1) and which we have listed in Appendix II, equation (1) is known or conjectured to have a period-k trichotomy for each $k \in \{3, 4, 5, 6\}$. These four trichotomies that we have listed in appendix III reveal the last group of new special cases of equation (1) which possess unbounded solutions in some range of their parameters. These are the following 12 equations:

$$\#8, \quad \#14, \quad \#15, \quad \#28^*, \quad \#35, \quad \#36,$$
$$\#44^*, \quad \#50, \quad \#51, \quad \#59^*, \quad \#70^*, \quad \#123^*$$

The boundedness character of the five equations with an asterisk has not yet been established. See also the table in Appendix I.

Open Problem 7 *For each of the five equations with an asterisk in the above list, determine the region of parameters where every solution of the equation is bounded.*

Note that there are also some thought provoking conjectures stated in the Appendices.

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Appendix I

This appendix, which summarizes the boundedness character of solutions of each of the 225 special cases of equation (1), is based on a thorough analysis of the existing literature, on numerous computer observations, and on many analytic investigations including the three new theorems, namely Theorem 2, Theorem 3, and Theorem 4, which are presented here.

A glance at any special case will immediately reveal whether the equation has unbounded solutions in some range of its parameters. At a glance we will also know whether the boundedness character of the equation is an established result or still a conjecture.

A bold faced **B** next to an equation in the table indicates that it is known that every solution of that equation is bounded. Next to the **B** we will also present a reference where the boundedness of the equation was established, unless the equation is of some simple form and the boundedness of all solutions is straightforward. For example, linear, Riccati, an equation where all corresponding terms of the numerator are also present in the denominator, etc. Similarly we print a bold faced **U** if it is known that the equation has unbounded solutions in some region of its parameters. Again we will also present a reference for all cases which are not straightforward.

A bold faced $\mathbf{B}^*$ next to an equation indicates that we only conjecture that every solution of the equation is bounded. Similarly, a bold faced $\mathbf{U}^*$ next to an equation indicates that we only conjecture that the equation has unbounded solutions in some range of its parameters.

Note that there are 31 equations in the table with a $\mathbf{B}^*$ next to them and 13 equations with a $\mathbf{U}^*$. That is, there are 44 special cases of equation (1) out of the 225 possible cases for which our conjecture about their boundedness remains to be confirmed or refuted (Table 1).

Appendix II

In this appendix, we present some known boundedness results for equation (1) when $C = 0$ and in particular all known period-two trichotomies for equation (1).

Table 1. Table of the boundedness character of the 225 equations.

#1	$x_{n+1}=\frac{\alpha}{A}, \quad n=0,1,\ldots$	**B**	
#2	$x_{n+1}=\frac{\alpha}{Bx_n}, \quad n=0,1,\ldots$	**B**	
#3	$x_{n+1}=\frac{\alpha}{Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#4	$x_{n+1}=\frac{\alpha}{Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#5	$x_{n+1}=\frac{\beta}{A}x_n, \quad n=0,1,\ldots$	**U**	
#6	$x_{n+1}=\frac{\beta}{B}, \quad n=0,1,\ldots$	**B**	
#7	$x_{n+1}=\frac{\beta x_n}{Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	([28] or Theorem 1)
#8	$x_{n+1}=\frac{\beta x_n}{Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	
#9	$x_{n+1}=\frac{\gamma}{A}x_{n-1}, \quad n=0,1,\ldots$	**U**	
#10	$x_{n+1}=\frac{\gamma x_{n-1}}{Bx_n}, \quad n=0,1,\ldots$	**U**	
#11	$x_{n+1}=\frac{\gamma}{C}, \quad n=0,1,\ldots$	**B**	
#12	$x_{n+1}=\frac{\gamma x_{n-1}}{Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	
#13	$x_{n+1}=\frac{\delta}{A}x_{n-2}, \quad n=0,1,\ldots$	**U**	
#14	$x_{n+1}=\frac{\delta x_{n-2}}{Bx_n}, \quad n=0,1,\ldots$	**U**	
#15	$x_{n+1}=\frac{\delta x_{n-2}}{Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	
#16	$x_{n+1}=\frac{\delta}{D}, \quad n=0,1,\ldots$	**B**	
#17	$x_{n+1}=\frac{\alpha}{A+Bx_n}, \quad n=0,1,\ldots$	**B**	
#18	$x_{n+1}=\frac{\alpha}{A+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#19	$x_{n+1}=\frac{\alpha}{A+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#20	$x_{n+1}=\frac{\alpha}{Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	([28], or [30], or Theorem 1)
#21	$x_{n+1}=\frac{\alpha}{Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	([16,19])
#22	$x_{n+1}=\frac{\alpha}{Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	([19] or [30])
#23	$x_{n+1}=\frac{\beta x_n}{A+Bx_n}, \quad n=0,1,\ldots$	**B**	
#24	$x_{n+1}=\frac{\beta x_n}{A+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	([28] or Theorem 1)
#25	$x_{n+1}=\frac{\beta x_n}{A+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	[27]
#26	$x_{n+1}=\frac{\beta x_n}{Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	([28] or Theorem 1)
#27	$x_{n+1}=\frac{\beta x_n}{Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#28	$x_{n+1}=\frac{\beta x_n}{Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{U}^*$	
#29	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Bx_n}, \quad n=0,1,\ldots$	**U**	([22], or [28], or Theorem 1)
#30	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#31	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([1,2])
#32	$x_{n+1}=\frac{\gamma x_{n-1}}{Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#33	$x_{n+1}=\frac{\gamma x_{n-1}}{Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([13])
#34	$x_{n+1}=\frac{\gamma x_{n-1}}{Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#35	$x_{n+1}=\frac{\delta x_{n-2}}{A+Bx_n}, \quad n=0,1,\ldots$	**U**	(Appendix III)
#36	$x_{n+1}=\frac{\delta x_{n-2}}{A+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Appendix III)
#37	$x_{n+1}=\frac{\delta x_{n-2}}{A+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#38	$x_{n+1}=\frac{\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Appendix III, or [9], or Theorem 4)
#39	$x_{n+1}=\frac{\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#40	$x_{n+1}=\frac{\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#41	$x_{n+1}=\frac{\alpha+\beta x_n}{A}, \quad n=0,1,\ldots$	**U**	
#42	$x_{n+1}=\frac{\alpha+\beta x_n}{Bx_n}, \quad n=0,1,\ldots$	**B**	
#43	$x_{n+1}=\frac{\alpha+\beta x_n}{Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	([27], or [28], or Theorem 1)
#44	$x_{n+1}=\frac{\alpha+\beta x_n}{Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{U}^*$	
#45	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{A}, \quad n=0,1,\ldots$	**U**	
#46	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{Bx_n}, \quad n=0,1,\ldots$	**U**	([22], or [28], or Theorem 1)
#47	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#48	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([13])

#49	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A}, \quad n = 0, 1, \ldots$	**U**	
#50	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Bx_n}, \quad n = 0, 1, \ldots$	**U**	([5])
#51	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	([3])
#52	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#53	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A}, \quad n = 0, 1, \ldots$	**U**	
#54	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Bx_n}, \quad n = 0, 1, \ldots$	**U**	([28] or Theorem 1)
#55	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([28] or Theorem 1)
#56	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U***	
#57	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A}, \quad n = 0, 1, \ldots$	**U**	
#58	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Bx_n}, \quad n = 0, 1, \ldots$	**B***	
#59	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U***	
#60	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	(can be transformed to #67)
#61	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A}, \quad n = 0, 1, \ldots$	**U**	
#62	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{Bx_n}, \quad n = 0, 1, \ldots$	**U**	([10], or [14], or Appendix II)
#63	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#64	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([12])
#65	$x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n}, \quad n = 0, 1, \ldots$	**B**	
#66	$x_{n+1} = \frac{\alpha+\beta x_n}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([27], or [28], or Theorem 1)
#67	$x_{n+1} = \frac{\alpha+\beta x_n}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	([27])
#68	$x_{n+1} = \frac{\alpha+\beta x_n}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([28] or Theorem 1)
#69	$x_{n+1} = \frac{\alpha+\beta x_n}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	([31] or Theorem 2)
#70	$x_{n+1} = \frac{\alpha+\beta x_n}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U***	(Appendix III)
#71	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([22], or [28], or Theorem 1)
#72	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	
#73	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([1,13])
#74	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([28] or Theorem 1)
#75	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([13])
#76	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	([18] or Theorem 2)
#77	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**B***	
#78	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#79	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#80	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#81	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	([15] or Theorem 2)
#82	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	([21] or Theorem 2)
#83	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([28])
#84	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([28] or Theorem 1)
#85	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U***	
#86	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	
#87	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([17] or Theorem 3)
#88	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#89	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**B***	
#90	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#91	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#92	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#93	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#94	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#95	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([10])
#96	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#97	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([12,24])
#98	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)

#99	$x_{n+1}=\frac{\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([6] or Theorem 3)
#100	$x_{n+1}=\frac{\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#101	$x_{n+1}=\frac{\alpha}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#102	$x_{n+1}=\frac{\alpha}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#103	$x_{n+1}=\frac{\alpha}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#104	$x_{n+1}=\frac{\alpha}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	([19,30])
#105	$x_{n+1}=\frac{\beta x_n}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#106	$x_{n+1}=\frac{\beta x_n}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#107	$x_{n+1}=\frac{\beta x_n}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	([27])
#108	$x_{n+1}=\frac{\beta x_n}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#109	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#110	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([13] or Theorem 3)
#111	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#112	$x_{n+1}=\frac{\gamma x_{n-1}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#113	$x_{n+1}=\frac{\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Appendix III, or [9], or Theorem 4)
#114	$x_{n+1}=\frac{\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#115	$x_{n+1}=\frac{\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#116	$x_{n+1}=\frac{\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#117	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A}, \quad n=0,1,\ldots$	**U**	
#118	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n}, \quad n=0,1,\ldots$	**U**	([28] or Theorem 1)
#119	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	([28] or Theorem 1)
#120	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{U}^*$	
#121	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{A}, \quad n=0,1,\ldots$	**U**	
#122	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{Bx_n}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#123	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{Cx_{n-1}}, \quad n=0,1,\ldots$	$\mathbf{U}^*$	
#124	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Can be transformed to #67)
#125	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A}, \quad n=0,1,\ldots$	**U**	
#126	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n}, \quad n=0,1,\ldots$	**U**	([2,10])
#127	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#128	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([12])
#129	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A}, \quad n=0,1,\ldots$	**U**	
#130	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n}, \quad n=0,1,\ldots$	**U**	([10])
#131	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#132	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{U}^*$	
#133	$x_{n+1}=\frac{\alpha}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#134	$x_{n+1}=\frac{\beta x_n}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#135	$x_{n+1}=\frac{\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#136	$x_{n+1}=\frac{\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#137	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A}, \quad n=0,1,\ldots$	**U**	
#138	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n}, \quad n=0,1,\ldots$	**U**	([10])
#139	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#140	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{U}^*$	
#141	$x_{n+1}=\frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#142	$x_{n+1}=\frac{\alpha+\beta x_n}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#143	$x_{n+1}=\frac{\alpha+\beta x_n}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#144	$x_{n+1}=\frac{\alpha+\beta x_n}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Theorem 2)
#145	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#146	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	([13] or Theorem 3)
#147	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#148	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Theorem 2)

#149	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#150	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#151	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#152	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	(Theorem 2)
#153	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	
#154	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 3)
#155	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#156	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#157	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#158	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#159	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#160	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#161	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#162	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([7] or Theorem 3)
#163	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#164	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#165	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([23], or [28], or Theorem 1)
#166	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([25] or Theorem 1)
#167	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U***	
#168	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B**	([25] or Theorem 1)
#169	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 3)
#170	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#171	$x_{n+1} = \frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**B***	
#172	$x_{n+1} = \frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#173	$x_{n+1} = \frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#174	$x_{n+1} = \frac{\alpha+\beta x_n+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#175	$x_{n+1} = \frac{\alpha+\beta x_n+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	(Theorem 2)
#176	$x_{n+1} = \frac{\alpha+\beta x_n+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#177	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([10])
#178	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#179	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	([12])
#180	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#181	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 3)
#182	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	(Theorem 2)
#183	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([10])
#184	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#185	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U***	
#186	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)
#187	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 3)
#188	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B***	
#189	$x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#190	$x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#191	$x_{n+1} = \frac{\alpha+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#192	$x_{n+1} = \frac{\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#193	$x_{n+1} = \frac{\beta x_n+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#194	$x_{n+1} = \frac{\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**B**	
#195	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n}, \quad n = 0, 1, \ldots$	**U**	([10])
#196	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**B***	
#197	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Dx_{n-2}}, \quad n = 0, 1, \ldots$	**U***	
#198	$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}}, \quad n = 0, 1, \ldots$	**U**	(Theorem 4)

#199	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	(Theorem 3)
#200	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#201	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**B**	
#202	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	(Theorem 3)
#203	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#204	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Theorem 2)
#205	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Theorem 4)
#206	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#207	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#208	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Theorem 2)
#209	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Theorem 4)
#210	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	(Theorem 3)
#211	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#212	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Theorem 2)
#213	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Theorem 4)
#214	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	(Theorem 3)
#215	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#216	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#217	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#218	$x_{n+1}=\frac{\alpha+\beta x_n+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#219	$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#220	$x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	
#221	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}}, \quad n=0,1,\ldots$	**U**	(Theorem 4)
#222	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Dx_{n-2}}, \quad n=0,1,\ldots$	**U**	(Theorem 3)
#223	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	$\mathbf{B}^*$	
#224	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	(Theorem 2)
#225	$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}, \quad n=0,1,\ldots$	**B**	

There exist 104 special cases of equation (1) that we know have only bounded solutions and 31 that we conjecture have only bounded solutions.
There exist 77 special cases of equation (1) that we know have unbounded solutions and 13 special cases that we conjecture have unbounded solutions.
If we confirm our conjectures there will be 135 special cases of equation (1) with only bounded solutions and 90 special cases that possess unbounded solutions.

The following period-two trichotomy results have recently been established for the following rational equations with nonnegative parameters and nonnegative initial conditions:

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+x_n}, \quad n=0,1,\ldots \tag{10}$$

$$x_{n+1}=\frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+x_{n-2}}, \quad n=0,1,\ldots \tag{11}$$

$$x_{n+1}=\frac{\alpha+\gamma x_{n-1}}{A+Bx_n+x_{n-2}}, \quad n=0,1,\ldots \tag{12}$$

THEOREM A (See [22,23,28]) *The following period-two trichotomy result holds for equation* (10)*:*

(a) *Every solution of equation* (10) *has a finite limit if and only if*

$$\gamma<\beta+A.$$

(b) *Every solution of equation* (10) *converges to a* (*not necessarily prime*) *period-two solution of equation* (10) *if and only if*

$$\gamma = \beta + A.$$

(c) *Equation* (10) *has unbounded solutions if and only if*

$$\gamma > \beta + A.$$

Theorem B (See [1,12,24]) *Assume that*

$$\gamma + \delta + A > 0.$$

Then the following period-two trichotomy result holds for equation (11)*:*

(a) *Every solution of equation* (11) *has a finite limit if and only if*

$$\gamma < \delta + A.$$

(b) *Every solution of equation* (11) *converges to a* (*not necessarily prime*) *period-two solution of equation* (11) *if and only if*

$$\gamma = \delta + A.$$

(c) *Equation* (11) *has unbounded solutions if and only if*

$$\gamma > \delta + A.$$

Theorem C (See [13]) *Assume that*

$$\gamma + A + B > 0.$$

Then the following period-two trichotomy result holds for equation (12)*:*

(a) *Every solution of equation* (12) *has a finite limit if and only if*

$$\gamma < A.$$

(b) *Every solution of equation* (12) *converges to a* (*not necessarily prime*) *period-two solution of equation* (12) *if and only if*

$$\gamma = A.$$

(c) *Equation* (12) *has unbounded solutions if and only if*

$$\gamma > A.$$

When, in addition to $C = 0$, we also assume that $D = 0$, the following general result was established in [10] for the rational equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots \tag{13}$$

with nonnegative parameters and nonnegative initial conditions.

Theorem D (See [10])

(a) *Assume that*

$$\gamma > \beta + \delta + A.$$

Then equation (13) *has unbounded solutions. More precisely, let k be any number such that*

$$0 < k < \gamma - \beta - \delta - A.$$

Then every solution of equation (13) *with initial conditions* x_{-2}, x_{-1}, x_0 *such that*

$$x_{-2}, x_0 \in (0, \gamma - A) \quad \textit{and} \;\; x_{-1} > \frac{\alpha + \gamma^2}{k + A}$$

is unbounded and in fact

$$\lim_{n\to\infty} x_{2n+1} = \infty \;\; \textit{and} \;\; \lim_{n\to\infty} x_{2n} = \frac{\beta\gamma + \delta A}{\gamma - \delta}.$$

(b) *Assume that*

$$\gamma = \beta + \delta + A \;\; \textit{and} \;\; \beta + A > 0. \tag{14}$$

Then every solution of equation (13) *converges to a* (*not necessarily prime*) *period-two solution and in particular all solutions of equation* (13) *are bounded.*

Without the assumption that

$$\beta + A > 0$$

in equation (13), it may not be true that every solution of equation (13) is bounded when

$$\gamma = \beta + \delta + a.$$

See equations (2,14).

Equation (13) does not have a trichotomy character in the spirit of Theorem A for equation (10). Actually it is not true that when

$$\gamma < \beta + A \tag{15}$$

holds, every solution of equation (13) has a finite limit. This is true when $\delta = 0$, but when $\delta > 0$,

$$\gamma < \beta + \delta + A \tag{16}$$

is not sufficient even for the local asymptotic stability of the equilibrium point of equation (13). However, we conjecture that when equation (16) holds every solution of equation (13) is bounded.

Appendix III

In this appendix, we present all special cases of equation (1) with period-3, period-4, period-5 and period-6 trichotomies which are known to us.

Period-3 Trichotomy (See [9,26]***)***

THEOREM E *Assume that*

$$A, B, C \in [0, \infty) \quad \textit{with} \;\; B + C > 0.$$

*Then the following period-*3 *trichotomy result is true for the rational equation*

$$x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots. \tag{17}$$

(a) *Every solution of equation* (17) *converges to* 0 *if and only if*

$$A > 1.$$

(b) *Every solution of equation* (17) *converges to a period-*3 *solution of the form*

$$\ldots, 0, 0, \phi, \ldots$$

with $\phi \geq 0$ *if and only if*

$$A = 1.$$

(c) *Equation* (17) *has unbounded solutions if and only if*

$$A < 1.$$

Note that the boundedness character of the equations

#14, #15, #35, #36, #38, #113

is covered by the period-3 trichotomy. See also the table in Appendix I.

Period-4 Trichotomy (See [8]***)***

CONJECTURE 2 *Assume that*

$$\alpha, \beta \in [0, \infty).$$

*then the following period-*4 *trichotomy result is true for the rational equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-2}}{x_{n-1}}, \quad n = 0, 1, \ldots. \tag{18}$$

(a) *Every solution of equation* (18) *converges to its positive equilibrium if and only if*

$$\beta > 1.$$

(b) *Every solution of equation* (18) *converges to a period-*4 *solution of equation* (18) *if and only if*

$$\beta = 1.$$

(c) *Equation* (18) *has unbounded solutions if and only if*

$$\beta < 1.$$

Note that the boundedness character of the equations

#51, #59*, #123*

is covered by the period-4 trichotomy. The boundedness character of each of the two equations with an asterisk above has not yet been established. See also the table in Appendix I.

Period-5 Trichotomy (See [5,8]***)***

Assume that

$$\alpha \geq 0.$$

Then the following period-5 trichotomy result is partially established and still partially a conjecture (as stated) for the rational equation

$$x_{n+1} = \frac{\alpha + x_{n-2}}{x_n}, \quad n = 0, 1, \ldots. \tag{19}$$

(a) *(Conjecture) Assume that*

$$\alpha > 1.$$

then every solution of the equation converges to its positive equilibrium point.

(b) *Assume that*

$$\alpha = 1.$$

Then every solution of equation (19) converges to a period-5 solution of equation (19).

(c) *Assume that*

$$\alpha < 1.$$

then equation (19) has unbounded solutions.

Note that the boundedness character of the equations

#14, #50

is covered by the period-5 trichotomy. See also the table in Appendix I.

Period-6 Trichotomy (See [8]***)***

Conjecture 2 *Assume that*

$$\alpha, C \in [0, \infty).$$

*Then the following period-*6 *trichotomy result is true for the rational equation*

$$x_{n+1} = \frac{\alpha + x_n}{Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots. \tag{20}$$

(a) *Every solution of equation* (20) *converges to its positive equilibrium if and only if*

$$\alpha C^2 > 1.$$

(b) *Every solution of equation* (20) *converges to a period*-6 *solution of equation* (11) *if and only if*

$$\alpha C^2 = 1.$$

(c) *Equation* (20) *has unbounded solutions if and only if*

$$\alpha C^2 < 1.$$

Note that the boundedness character of the equations

$$\#28, \quad \#44, \quad \#70$$

is covered by the period-6 trichotomy conjecture. The boundedness character of each of the three equations above has not been established yet. See also the table in Appendix I.