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Patterns of boundedness of a rational system in the plane

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We investigate the boundedness character of non-negative solutions of a rational system in the plane. The system contains 10 parameters with non-negative real values and consists of 343 special cases, each with positive parameters. In 342 out of the 343 special cases, we establish easily verifiable necessary and sufficient conditions, explicitly stated in terms of 10 parameters, which determine the boundedness character of solutions of the system. In the remaining special case, we conjecture the boundedness character of solutions. It is interesting to note that this special case can be transformed to the well-known May's Host-Parasitoid model.

Keywords: boundedness; global stability; patterns of boundedness; rational equations; rational systems

AMS Subject Classification: 39A10

1. Introduction

We establish the boundedness character of solutions of the rational system in the plane,

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{1.1}$$

with non-negative parameters and arbitrary non-negative initial conditions such that the denominators are always positive and such that

$$A_1 + B_1 + \alpha_2 + B_2 + C_2 > 0. \tag{1.2}$$

System (1.1) contains

$$1 \times (2^3 - 1) \times (2^3 - 1) \times (2^3 - 1) = 343,$$

special cases of systems, each with positive parameters. In one special case, all the parameters of the system are positive. In the remaining 342 special cases, at least one of the 10 parameters of System (1.1) is allowed to be zero.

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In Appendix 1, we define the following four possible boundedness characterizations:

$$(\mathrm{B},\mathrm{B}),\quad (\mathrm{B},\mathrm{U}),\quad (\mathrm{U},\mathrm{B}),\quad \text{and}\quad (\mathrm{U},\mathrm{U}).$$

When each special case, within a group of special cases of System (1.1), has the boundedness characterization (B, B) (respectively, (B, U), (U, B), and (U, U)), then we will say that System (1.1), restricted to this group of special cases, has the boundedness characterization (B, B) (respectively, (B, U), (U, B), and (U, U)).

We establish easily verifiable necessary and sufficient conditions, explicitly stated in terms of the 10 parameters, under which the boundedness characterization of the system is: (B, B), (B, U), (U, B), or (U, U).

The boundedness character of solutions of a system is one of the main ingredients in understanding the global behaviour of the system including its global stability. Boundedness is also essential in the study of most applications. Actually, some special cases of System (1.1) arise in applications. For example, each of the following two systems of rational difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{x_n}{y_n} \\ y_{n+1} &= \alpha_2 + x_n \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{1.3}$$

and

$$\left.\begin{aligned} x_{n+1} &= \frac{x_n}{A_1 + y_n} \\ y_{n+1} &= x_n \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{1.4}$$

is reducible to the so-called Pielou's equation, that is, the equation

$$x_{n+1} = \frac{\beta x_n}{1 + x_{n-1}}, \quad n = 0, 1, \ldots$$

with $\beta \in (0, \infty)$ (see [8,9,13,14,19,20]).

The system of rational difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} &= \frac{\gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{1.5}$$

was studied in [12], as a discrete competition model for the populations x_n and y_n. See also [7,10,15] and the references cited therein.

Also, the system

$$\left.\begin{aligned} x_{n+1} &= \frac{x_n}{y_n} \\ y_{n+1} &= \frac{1}{\alpha} x_n + \frac{1}{\alpha} y_n \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{1.6}$$

with $\alpha \in (0, \infty)$, through the change of variables,

$$x_n = Y_n \quad \text{and} \quad y_n = \frac{1 + Y_n}{X_n}, \tag{1.7}$$

becomes

$$\left.\begin{aligned} X_{n+1} &= \frac{\alpha X_n}{1 + Y_n} \\ Y_{n+1} &= \frac{X_n Y_n}{1 + Y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{1.8}$$

which is the well-known May's Host-Parasitoid Model (see [16–18]).

System (1.6) is the only special case among the 343 special cases of System (1.1) whose boundedness characterization has not been established yet. For this case, we offer the following conjecture.

CONJECTURE 1.1. *For every positive solution* $\{x_n, y_n\}$ *of System* (1.6)*, the component* $\{x_n\}$ *is bounded for all positive values of the parameters, and in some range of the parameters and for some initial conditions, the component* $\{y_n\}$ *is unbounded.*

System (1.1) is a special case of the 'full' rational system in the plane,

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{1.9}$$

which contains

$$7 \times 7 \times 7 \times 7 = 2401$$

special cases each with positive parameters. A large number of open problems and conjectures about System (1.9) were posed in [7,10]. For some work on the boundedness character of System (1.9), see [1–6,10–11]. For the numbering system of the 2401 special cases contained in System (1.9), see Appendices 1 and 2 in [7].

Throughout this paper, we assume that (1.2) holds and we present necessary and sufficient conditions under which System (1.1) has the boundedness characterization (B, B), (B, U), (U, B), or (U, U).

When Condition (1.2) is satisfied, System (1.1) contains exactly 342 special cases of systems each with positive parameters. This is because when

$$A_1 + B_1 + \alpha_2 + B_2 + C_2 = 0,$$

System (1.1) reduces to the single special case (1.6).

First, we present the boundedness patterns of System (1.1) when

$$B_1 = 0,$$

that is, for the system

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots. \tag{1.10}$$

System (1.10) has the boundedness characterization (B, B), if and only if:

$$B_2 = 0, \quad C_1, \beta_2 \in (0, \infty), \quad \text{and} \quad (\gamma_2 = 0 \quad \text{or} \quad C_2 > 0). \tag{1.11}$$

Under Condition (1.11), System (1.10), according to the numbering system in [7], consists of the following 20 special cases:

$$\left.\begin{array}{l} (6,7), \ (6,8), \ (6,16), \ (6,22), \ (6,23), \\ (6,26), \ (6,31), \ (6,34), \ (6,41), \ (6,46), \\ (14,7), \ (14,8), \ (14,16), \ (14,22), \ (14,23), \\ (14,26), \ (14,31), \ (14,34), \ (14,41), \ (14,46) \end{array}\right\}. \tag{1.12}$$

System (1.10), restricted to the group of special cases (1.12), has the boundedness characterization (B, B). See also Appendix 1.

System (1.10) has the boundedness characterization (B, U), if and only if:

$$B_2 = C_2 = 0 \quad \text{and} \quad C_1, \beta_2, \gamma_2 \in (0, \infty). \tag{1.13}$$

System (1.10) has the boundedness characterization (U, B), if and only if:

$$(\beta_2 = 0 \quad \text{or} \quad B_2 > 0), \quad (\gamma_2 = 0 \quad \text{or} \quad C_2 > 0), \quad \text{and} \quad A_2 + C_2 > 0. \tag{1.14}$$

When none of the above three conditions (1.11), (1.13), and (1.14) is satisfied, System (1.10) has the boundedness characterization (U, U).

One can see that System (1.10) contains 146 special cases of which 20 special cases have the boundedness characterization (B, B), 3 special cases have the boundedness characterization (B, U), 75 special cases have the boundedness characterization (U, B), and 48 special cases have the boundedness characterization (U, U). See Appendix 1.

Next, we state necessary and sufficient conditions that describe the boundedness patterns of System (1.1) when

$$B_1 > 0. \tag{1.15}$$

System (1.1), with $B_1 > 0$, has the boundedness characterization (B, B), if and only if, one of the following two conditions is satisfied:

$$C_2 \in (0, \infty) \tag{1.16}$$

or

$$\begin{aligned} &A_1 + C_1 > 0 \ \ \gamma_2 = C_2 = 0 \quad \text{and} \quad (\alpha_2 = 0 \quad \text{or} \quad A_2 > 0). \\ &A_1 = C_1 = \gamma_2 = C_2 = 0 \end{aligned} \tag{1.17}$$

When none of the above two conditions (1.16) and (1.17) is satisfied, System (1.1), with $B_1 > 0$, has the boundedness characterization (B, U).

One can see that System (1.1), with $B_1 > 0$, contains 196 special cases of which 142 special cases have the boundedness characterization (B, B) and 54 special cases have the boundedness characterization (B, U). We should mention that when $B_1 > 0$, none of the special cases of System (1.1) has the boundedness characterization (U, B) or (U, U). See Appendix 2.

In Section 2, we establish that Condition (1.11) is necessary and sufficient for every solution of System (1.10) to be bounded. When (1.15) holds, the proof is much simpler, and the details are omitted.

We should point out to the reader that in Section 2 of this paper, when we investigate a special case of System (1.10), for simplicity and convenience we may write the system in normalized form, by using an appropriate change of variables of the form:

$$x_n = \lambda X_n \quad \text{and} \quad y_n = \mu Y_n.$$

By doing so, some of the parameters of the system may be chosen equal to one.

2. Necessary and sufficient conditions for the boundedness of solutions of system (1.10)

The main result in this section is the following theorem, whose proof is long and will be subdivided into several lemmas and observations.

THEOREM 2.1. *Assume that Condition* (1.2) *holds. Then every solution of System* (1.10) *is bounded, if and only if, Condition* (1.11) *is satisfied.*

Assume that Conditions (1.2) and (1.11) are satisfied. We will show that every solution of System (1.10) is bounded. As we can see, among the 146 special cases of System (1.10), the 20 special cases listed in (1.12) are the only special cases for which Condition (1.11) is satisfied.

The proof of boundedness of solutions of the first seven special cases of the systems listed in (1.12) is straightforward. Actually, in the special case (6, 8), every solution $\{x_n, y_n\}$ of the system is constant, for $n \geq 2$, and in the remaining six cases the component $\{x_n\}$ of the solution satisfies a second-order rational difference equation for which it is known from [9] that every solution is bounded. The component $\{y_n\}$ in each of these special cases is also clearly bounded.

The next lemma establishes the boundedness character of solutions of the special case (6, 34).

LEMMA 2.1. *Assume that*

$$\beta_2, \gamma_2, A_2 \in (0, \infty).$$

Then, every solution of the system

$$(6,34): \left.\begin{aligned} x_{n+1} &= \frac{x_n}{y_n} \\ y_{n+1} &= \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{2.1}$$

is bounded.

Proof. Let $\{x_n, y_n\}$ be a solution of System (2.1). Note that

$$x_{n+2} = \frac{x_{n+1}}{y_{n+1}} = \frac{x_n}{\beta_2 x_n + \gamma_2 y_n}\left(1 + \frac{A_2}{y_n}\right), \quad \text{for } n \geq 0, \tag{2.2}$$

$$y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} < \frac{\beta_2 x_n}{y_n} + \gamma_2 = \beta_2 x_{n+1} + \gamma_2, \quad \text{for } n \geq 0, \tag{2.3}$$

and by eliminating the component $\{y_n\}$ from System (2.1), we find

$$x_{n+1} = \frac{x_n}{x_{n-1}} \cdot \frac{A_2 x_n + x_{n-1}}{\beta_2 x_n + \gamma_2}, \quad n = 0, 1 \ldots. \tag{2.4}$$

Now, the following two observations will be useful in the proof.

(1) In view of (2.2),

$$x_{n+2} \to \infty \Rightarrow y_n \to 0, \tag{2.5}$$

or equivalently,

$$\text{if } \{y_n\} \text{ is bounded from below} \Rightarrow \{x_n\} \text{ is bounded.}$$

(2) From the second equation of the system

$$y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n},$$

we see that

$$\{x_n\} \text{ is bounded from below} \Rightarrow \{y_n\} \text{ is bounded from below.} \tag{2.6}$$

Also, from (2.2), we see that

$$x_{n+2} \to 0 \Rightarrow \frac{x_n}{\beta_2 x_n + \gamma_2 y_n} = \frac{\frac{x_n}{y_n}}{\frac{\beta_2 x_n}{y_n} + \gamma_2} \to 0$$

and so

$$x_{n+1} = \frac{x_n}{y_n} \to 0.$$

In general, we have that

$$x_{n_i} \to 0 \Rightarrow x_{n_i - k} \to 0, \quad \text{for all} \quad k \geq 1. \tag{2.7}$$

From (2.4), we have

$$\frac{x_{n+1}}{x_n} = \frac{1}{x_{n-1}} \cdot \frac{A_2 x_n + x_{n-1}}{\beta_2 x_n + \gamma_2} < \frac{A_2}{\gamma_2} \cdot \frac{x_n}{x_{n-1}} + \frac{1}{\gamma_2}, \quad \text{for } n \geq 0,$$

and so when

$$A_2 < \gamma_2,$$

the quotient x_{n+1}/x_n is bounded. But

$$\frac{x_{n+1}}{x_n} = \frac{1}{y_n},$$

which implies that the component $\{y_n\}$ is bounded from below. Therefore, from (2.5) it follows that $\{x_n\}$ is bounded, which in view of (2.3) implies that the component $\{y_n\}$ is also bounded.

To complete the proof, assume that

$$A_2 \geq \gamma_2.$$

In view of (2.3), (2.5), and (2.6), it suffices to show that the component $\{x_n\}$ of the solution is bounded from below.

Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \longrightarrow 0 \quad \text{and} \quad x_{n_i+1} < x_n, \quad \text{for } n < n_i + 1.$$

In view of (2.7), we have

$$x_{n_i} \longrightarrow 0,$$

and also

$$x_{n_i-j} \longrightarrow 0, \quad \text{for all} \quad j \geq 0.$$

Choose a positive integer m such that

$$\left(\frac{\gamma_2}{A_2}\right)^m \gamma_2 < 1 \quad \text{and} \quad m > \gamma_2. \tag{2.8}$$

There exists a positive number ϵ, sufficiently small, such that

$$x_{n_i-j} < \epsilon < \max\left\{\gamma_2, \frac{\gamma_2}{\beta_2}\right\}, \quad \text{for all } j \in \{0, \ldots, m\}.$$

We now claim that, eventually

$$y_{n_i} \leq 1. \tag{2.9}$$

To prove (2.9), we consider the following three cases:

Case 1.

$$\gamma_2 \leq 1.$$

Then,

$$y_{n_i} = \frac{\beta_2 x_{n_i-1} + \gamma_2 y_{n_i-1}}{A_2 + y_{n_i-1}} \leq \frac{A_2 + y_{n_i-1}}{A_2 + y_{n_i-1}} = 1.$$

Case 2.

$$1 < \gamma_2 < A_2.$$

Set

$$l_{-j} = \liminf_{n\to\infty} y_{n_i-j}, \quad \text{for all } j \geq 0.$$

Then,

$$y_{n_i-j} = \frac{\beta_2 x_{n_i-j-1} + \gamma_2 y_{n_i-j-1}}{A_2 + y_{n_i-j-1}} < \frac{\gamma_2 + \gamma_2 y_{n_i-j-1}}{1 + y_{n_i-j-1}} = \gamma_2,$$

and so it follows that

$$l_{-j} \leq \gamma_2, \quad \text{for all } j \geq 0.$$

Then, in view of (2.8)

$$l_0 = \frac{\gamma_2}{A_2 + l_{-1}} l_{-1} \leq \frac{\gamma_2}{A_2} l_{-1} \leq \left(\frac{\gamma_2}{A_2}\right)^2 l_{-2} \leq \cdots \leq \left(\frac{\gamma_2}{A_2}\right)^m l_{-m} \leq \left(\frac{\gamma_2}{A_2}\right)^m \gamma_2 < 1,$$

which implies that, eventually (2.9) is true.

Case 3.

$$1 < \gamma_2 = A_2.$$

From

$$l_{-j} = \frac{\gamma_2 l_{-j-1}}{\gamma_2 + l_{-j-1}}, \quad j = 0, 1, \ldots,$$

we see that

$$l_0 = \frac{\gamma_2 l_{-m}}{\gamma_2 + m l_{-m}},$$

and so in view of (2.8), we have that

$$l_0 < 1,$$

which implies that, eventually (2.9) is true.

Hence, eventually,

$$x_{n_i+1} = \frac{x_{n_i}}{y_{n_i}} \geq x_{n_i},$$

which is a contradiction and the proof is complete. □

LEMMA 2.2. *In each of the following nine special cases:* (*which are listed in* (1.12))

$$(14, 7),\ \ (14, 8),\ \ (14, 16),\ \ (14, 22),$$

$$(14, 26),\ \ (14, 31),\ \ (14, 34),\ \ (14, 41),\ \ (14, 46),$$

every solution is bounded.

Proof. The proof of boundedness for the special case (14, 8) is trivial. The remaining eight special cases are included in the system, which in normalized form is written as follows:

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{1 + y_n} \\ y_{n+1} &= \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{2.10}$$

with

$$A_2 + \gamma_2 > 0 \quad \text{and} \quad (\gamma_2 = 0 \quad \text{or} \quad C_2 > 0).$$

Clearly the quotient,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\beta_1 x_n}{\alpha_2 + x_n + \gamma_2 y_n} \cdot \frac{A_2 + C_2 y_n}{1 + y_n}, \quad \text{for all } n \geq 0,$$

is bounded from above by the positive number M

$$M = \begin{cases} \beta_1 \max\{A_2, C_2\} & \text{if } A_2 > 0 \text{ and } C_2 > 0, \\ \beta_1 A_2 & \text{if } A_2 > 0 \text{ and } C_2 = 0, \\ \beta_1 C_2 & \text{if } A_2 = 0 \text{ and } C_2 > 0. \end{cases}$$

Hence,

$$x_{n+1} = \frac{\beta_1 x_n}{1 + y_n} < \beta_1 \cdot \frac{x_n}{y_n} < \beta_1 M, \quad \text{for } n \geq 1,$$

and so the component $\{x_n\}$ of every solution is bounded. Then, eventually,

$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + C_2 y_n} + \frac{x_n}{A_2 + C_2 y_n} \leq \frac{\max(\alpha_2, \gamma_2)}{\min(A_2, C_2)} + \frac{1}{C_2} \cdot \frac{x_n}{y_n} \leq \frac{\max(\alpha_2, \gamma_2)}{\min(A_2, C_2)} + \frac{M}{C_2},$$

provided that $A_2 > 0$ and $C_2 > 0$, or

$$y_{n+1} \leq \frac{\gamma_2}{C_2} + \frac{M}{C_2},$$

provided that $\alpha_2 = A_2 = 0$ and $C_2 > 0$, or finally

$$y_{n+1} \leq \frac{\alpha_2 + \beta_1 M}{A_2},$$

provided that $\gamma_2 = C_2 = 0$ and $A_2 > 0$. Therefore, the component $\{y_n\}$ of every solution is also bounded. The proof is complete. □

The next lemma establishes the boundedness of solutions in the special case

$$(14, 23): \left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{1 + y_n} \\ y_{n+1} &= \frac{\alpha_2 + x_n}{y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots. \tag{2.11}$$

LEMMA 2.3. *Assume that*

$$\beta_1, \alpha_2 \in (0, \infty).$$

Then every solution of System (2.11) *is bounded.*

Proof. Let $\{x_n, y_n\}$ be a solution of System (2.11). Clearly,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\beta_1 x_n}{\alpha_2 + x_n} \cdot \frac{y_n}{1 + y_n} \leq \beta_1, \quad \text{for all } n \geq 0,$$

and so

$$x_{n+1} = \frac{\beta_1 x_n}{1 + y_n} \leq \beta_1 \cdot \frac{x_n}{y_n} \leq \beta_1^2, \quad \text{for all } n \geq 1.$$

Hence, the component $\{x_n\}$ of the solution $\{x_n, y_n\}$ is bounded. Now, we claim that the component $\{y_n\}$ of the solution is also bounded. Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that, as $i \longrightarrow \infty$,

$$y_{n_i+1} \longrightarrow \infty \quad \text{and} \quad y_{n_i+1} > y_n, \quad \text{for all } n < n_i + 1. \tag{2.12}$$

From

$$y_{n_i+1} = \frac{\alpha_2 + x_{n_i}}{\alpha_2 + x_{n_i-1}} y_{n_i-1},$$

and in view of (2.12), we see that eventually,

$$x_{n_i} > x_{n_i-1}.$$

Then

$$x_{n_i} = \frac{\beta_1 x_{n_i - 1}}{1 + y_{n_i - 1}} > x_{n_i - 1},$$

which implies that eventually

$$y_{n_i - 1} < \beta_1 - 1. \tag{2.13}$$

On the other hand,

$$y_{n_i + 1} = \frac{\alpha_2}{y_{n_i}} + \frac{x_{n_i}}{y_{n_i}} \longrightarrow \infty,$$

which implies that

$$y_{n_i} = \frac{\alpha_2}{y_{n_i - 1}} + \frac{x_{n_i - 1}}{y_{n_i - 1}} \longrightarrow 0,$$

from which it follows that

$$y_{n_i - 1} \longrightarrow \infty.$$

This contradicts (2.13) and completes the proof. □

It is interesting to note that, for all positive values of the parameters, System (2.11) possesses the non-hyperbolic equilibrium point

$$(0, \sqrt{\alpha_2}), \tag{2.14}$$

and the prime period-two cycle

$$\ldots, (0, y_0), \ \left(0, \frac{\alpha_2}{y_0}\right), \ (0, y_0), \ \left(0, \frac{\alpha_2}{y_0}\right), \ldots. \tag{2.15}$$

Also, when

$$\beta_1 > 1 + \sqrt{\alpha_2}, \tag{2.16}$$

System (2.11) possesses the unique positive equilibrium solution

$$((\beta_1 - 1)^2 - \alpha_2, \beta_1 - 1), \tag{2.17}$$

which is locally asymptotically stable, as long as (2.16) is satisfied.

For the global behaviour of solutions of System (2.11), we offer the following conjecture. For some results on the global character of solutions of rational systems in the plane, see [1,2,7–10].

Conjecture 2.1. *Every solution of System* (2.11) *converges to a* (*not necessarily prime*) *period-two solution.*

The following theorem confirms Conjecture 2.1 in a special case.

Theorem 2.2. *Assume that*

$$\beta_1 \leq 1 + \sqrt{\alpha_2}. \tag{2.18}$$

Then every positive solution of System (2.11) *converges to a* (*not necessarily prime*) *period-two solution of the form* (2.15).

Proof. Let $\{x_n, y_n\}$ be a positive solution of System (2.11). It suffices to show that the component $\{x_n\}$ converges to zero.

Observe that

$$y_n + y_{n-1} = \frac{\alpha_2}{y_{n-1}} + y_{n-1} + \frac{x_{n-1}}{y_{n-1}} > \frac{\alpha_2}{y_{n-1}} + y_{n-1} \geq 2\sqrt{\alpha_2},$$

and

$$y_n y_{n-1} = \alpha_2 + x_{n-1} > \alpha_2.$$

Then

$$x_{n+1} = \frac{\beta_1 x_n}{1+y_n} = \frac{\beta_1^2 x_{n-1}}{(1+y_n)(1+y_{n-1})} = \frac{\beta_1^2 x_{n-1}}{1+y_n+y_{n-1}+y_n y_{n-1}},$$

implies that

$$x_{n+1} \leq \frac{\beta_1^2}{1+2\sqrt{\alpha_2}+\alpha_2} x_{n-1} = \left(\frac{\beta_1}{1+\sqrt{\alpha_2}}\right)^2 x_{n-1},$$

from which the result follows. □

Next, we establish the boundedness of solutions in the remaining two special cases which are listed in (1.12):

$$(6, 41) \text{ and } (6, 46).$$

The change of variables

$$y_n = \gamma_2 + Y_n,$$

transforms system (6, 41) to a system of the form (14, 31), whose boundedness was established in Lemma 2.2. Finally, the following lemma establishes the boundedness of solutions of the special case (6, 46).

Lemma 2.4. *Assume that*

$$\beta_1, \alpha_2, \beta_2, \gamma_2, A_2 \in (0, \infty).$$

Then, every solution of the system

$$(6,46):\quad \left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{2.19}$$

is bounded.

Proof. Let $\{x_n, y_n\}$ be a solution of System (2.19). Clearly,

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \geq \frac{\min\{\alpha_2, \gamma_2\}}{\max\{A_2, 1\}},$$

and so the component $\{y_n\}$ of the solution is bounded from below by the positive number

$$m = \frac{\min\{\alpha_2, \gamma_2\}}{\max\{A_2, 1\}}.$$

From this and in view of

$$\frac{x_{n+2}}{\beta_1} = \frac{x_{n+1}}{y_{n+1}} = \frac{\beta_1 x_n}{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}\left(\frac{A_2}{y_n} + 1\right) \leq \frac{\beta_1}{\beta_2}\left(\frac{A_2}{m} + 1\right), \quad \text{for all } \ n \geq 1,$$

we see that the component $\{x_n\}$ of the solution is bounded. By the second equation of the system, it follows that the component $\{y_n\}$ of the solution is also bounded and the proof is complete.

Next, we establish that Condition (1.11) is necessary for every solution of System (1.10) to be bounded. That is, we will show that when

$$C_1 = 0, \tag{2.20}$$

or when

$$C_1 > 0 \quad \text{and} \quad \beta_2 = 0, \tag{2.21}$$

or when

$$C_1 > 0, \quad \beta_2 > 0, \quad \text{and} \quad B_2 > 0, \tag{2.22}$$

or when

$$C_1 > 0, \quad \beta_2 > 0, \quad \gamma_2 > 0, \quad \text{and} \quad B_2 = C_2 = 0, \tag{2.23}$$

System (1.10) has unbounded solutions in a certain region of the parameters and for some initial conditions. In fact, we will prove that in each of the 126 special cases that correspond to Conditions (2.20)–(2.23), the component $\{x_n\}$ or the component $\{y_n\}$, of each solution, is unbounded in a certain region of the parameters and for some initial conditions. More precisely, we will obtain the boundedness characterization of each one of the 126 special cases. This will complete the proof that Condition (1.11) is a necessary and sufficient condition for every solution of System (1.10) to be bounded.

First, observe that when (2.20) holds, the component $\{x_n\}$ of the solution of System (1.10) is given by,

$$x_n = \left(\frac{\beta_1}{A_1}\right)^n x_0, \quad \text{for } n \geq 0.$$

Hence, when

$$\beta_1 > A_1 \quad \text{and} \quad x_0 > 0,$$

$$x_n \longrightarrow \infty,$$

and System (1.10) has unbounded solutions. More precisely, when (2.20) holds System (1.10) has the boundedness characterization (U, B) in 25 special cases and the boundedness characterization (U, U) in 24 special cases (see Appendix 1).

Next, assume that (2.21) holds. In this case, the second equation of the system becomes

$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}, \quad n = 0, 1, \ldots.$$

When

$$A_2 > 0 \quad \text{and} \quad C_2 = 0,$$

System (1.10) contains 12 special cases, of which the 8 special cases with

$$\gamma_2 > 0,$$

have the boundedness characterization (U, U) and the four special cases with

$$\gamma_2 = 0,$$

have the boundedness characterization (U, B). The proof is straightforward and the details will be omitted.

When

$$A_2 > 0 \quad \text{and} \quad C_2 > 0,$$

or when

$$A_2 = 0 \quad \text{and} \quad B_2 = 0,$$

we see that the component $\{y_n\}$ of every solution is bounded. Then the first equation of the system implies that, for β_1 sufficiently large, the component $\{x_n\}$ is unbounded. That is, in 18 additional special cases, System (1.10) has the boundedness characterization (U, B).

When

$$A_2 = 0 \quad \text{and} \quad B_2 > 0,$$

we claim that the system

$$\left.\begin{aligned} x_{n+1} &= \frac{x_n}{A_1 + y_n} \\ y_{n+1} &= \frac{\alpha_2 + \gamma_2 y_n}{x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{2.24}$$

with non-negative parameters has unbounded solutions.

When

$$C_2 > 0,$$

that is, in six special cases, the boundedness characterization of System (1.10) is (U, B). More precisely, when

$$A_1 > 0 \quad \text{and} \quad C_2 > 0,$$

the proof is given in [3]. When

$$A_1 = 0 \quad \text{and} \quad C_2 > 0,$$

that is, for the system

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{y_n} \\ y_{n+1} &= \frac{\alpha_2 + \gamma_2 y_n}{x_n + y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots,$$

one can see that the component $\{x_n\}$ of the solution satisfies the second-order rational equation

$$x_{n+1} = \frac{\beta_1 x_n (\beta_1 + x_n) x_{n-1}}{\alpha_2 x_n + \beta_1 \gamma_2 x_{n-1}}, \quad n = 0, 1, \ldots$$

and the result follows by employing Theorem 1.6 in [1]. Clearly, one can see that the component $\{y_n\}$ of the solution is bounded.

On the other hand, when

$$C_2 = 0,$$

the component $\{y_n\}$ of the solution $\{x_n, y_n\}$ of System (2.24) satisfies the second-order rational equation

$$y_{n+1} = \frac{y_n(\alpha_2 + \gamma_2 y_n)(A_1 + y_{n-1})}{\alpha_2 + \gamma_2 y_{n-1}}, \quad n = 0, 1, \ldots,$$

which is easily shown to have unbounded solutions (see also [3]). We should mention that the boundedness characterization of the system in this case is (U, U).

Next, assume that (2.22) holds. In this case, the system can be written in normalized form as follows:

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + y_n} \\ y_{n+1} &= \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{2.25}$$

with non-negative parameters.

When

$$A_2, C_2 \in (0, \infty),$$

or when

$$C_2 > 0 \quad \text{and} \quad \alpha_2 = A_2 = 0,$$

the component$\{y_n\}$ of the solution is clearly bounded, and so, for β_1 sufficiently large, the component $\{x_n\}$ of the solution is unbounded. That is, in 12 additional special cases, System (1.10) has the boundedness characterization (U, B).

When

$$\alpha_2, C_2 \in (0, \infty) \quad \text{and} \quad \gamma_2 = A_2 = 0,$$

that is for the two special cases, namely, (6, 33) and (14, 33) contained in

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n}{x_n + y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots,$$

the boundedness characterization is (U, B). The proof for the special case (14, 33) is given in [4]. The proof for the special case (6, 33) is similar.

When

$$\alpha_2, \gamma_2, C_2 \in (0, \infty) \quad \text{and} \quad A_2 = 0,$$

that is for the two special cases, namely, (6, 48) and (14, 48) contained in

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots,$$

the boundedness characterization is (U, B). The proof is straightforward and will be omitted.

When

$$C_2 = 0,$$

System (1.10) contains 16 special cases in each of which the component $\{x_n\}$ of the solution satisfies the second-order rational equation

$$x_{n+1} = \frac{\beta_1 x_n (A_2 + x_{n-1})}{A_1 A_2 + \alpha_2 + (A_1 + 1)x_{n-1}}, \quad n = 0, 1, \ldots$$

which in view of Theorem 0.6 in [1] has unbounded solutions. When

$$\gamma_2 = 0 \quad \text{and} \quad A_2 > 0,$$

or when

$$\alpha_2 = A_2 = \gamma_2 = 0,$$

that is, in six special cases, one can see that the component $\{y_n\}$ of the solution is bounded and so their boundedness characterization is (U, B). In the remaining 10 special cases, clearly the component $\{y_n\}$ of the solution is unbounded in some range of the parameters from which it follows that their boundedness characterization is (U, U).

Finally, assume that (2.23) holds. In this case, the system can be written in normalized form as follows:

$$\left.\begin{aligned} x_{n+1} &= \frac{x_n}{A_1 + y_n} \\ y_{n+1} &= \alpha_2 + \beta_2 x_n + \gamma_2 y_n \end{aligned}\right\}, \quad n = 0, 1, \ldots, \tag{2.26}$$

with

$$\beta_2, \gamma_2 \in (0, \infty), \quad A_1, \alpha_2 \in [0, \infty), \quad \text{and} \quad A_1 + \alpha_2 \in (0, \infty).$$

System (2.26) contains the following three cases:

$$(6, 40), \quad (14, 25) \quad \text{and} \quad (14, 40).$$

In each of these three cases, the boundedness characterization of the system is (B, U) as the following theorem shows.

THEOREM 2.3. *Let* $\{x_n, y_n\}$ *be a positive solution of System* (2.26). *The following statements are true:*

(*a*) *Assume that*

$$\beta_2 \in (0, \infty) \quad \text{and} \quad \gamma_2 \in [1, \infty).$$

Then, the solution $\{x_n, y_n\}$ *of System* (2.26) *satisfies*

$$\lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} y_n = \infty.$$

(*b*) *Assume that*

$$\beta_2 \in (0, \infty) \quad \text{and} \quad \gamma_2 \in (0, 1).$$

Then, the solution $\{x_n, y_n\}$ *of System* (2.26) *is bounded.*

Proof. Let $\{x_n, y_n\}$ be a positive solution of System (2.26). The proof of (a) follows from the fact that

$$y_{n+1} \geq \gamma_2 y_n, \quad \text{for all} \ \ n \geq 0.$$

(b) For first, we will show that the component $\{x_n\}$ of the solution is bounded. Observe that for all $n \geq 0$

$$x_{n+2} = \frac{x_{n+1}}{A_1 + y_{n+1}} = \frac{x_n}{A_1 + \alpha_2 + \beta_2 x_n + \gamma_2 y_n} \cdot \frac{1}{A_1 + y_n}. \tag{2.27}$$

When

$$A_1 > 0,$$

we find that

$$x_{n+2} \leq \frac{1}{\beta_2 A_1}, \quad \text{for all} \ \ n \geq 0,$$

and so the component $\{x_n\}$ of the solution is bounded. From the second equation of the system, it follows that the component $\{y_n\}$ is also bounded.

When

$$A_1 = 0,$$

(2.27) implies that

$$x_{n+2} \longrightarrow \infty \Rightarrow y_n \longrightarrow 0.$$

Assume for the sake of contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$x_{n_i+1} \longrightarrow \infty.$$

Then, clearly

$$y_{n_i-1} \longrightarrow 0,$$

which is a contradiction because $\{y_n\}$ is bounded from below by the positive constant α_2. Hence, the component $\{x_n\}$ of the solution is bounded. From the second equation of the system, it follows that the component $\{y_n\}$ of the solution is also bounded.

The proof of Theorem 2.1 is complete. □

Notes

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Appendix 1

The boundedness character of the rational system:

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots . \tag{2.28}$$

The boundedness characterization (B, B), next to a special case of System (2.28), means that both components of every solution of the system, in this special case, are bounded.

The boundedness characterization (B, U), next to a special case of System (2.28), means that the first component of every solution in this special case of the system is always bounded and there exist solutions in which the second component is unbounded in some range of the parameters and for some initial conditions.

The boundedness characterization (U, B), next to a special case of System (2.28), means that the second component of every solution in this special case of the system is always bounded and there exist solutions in which the first component is unbounded in some range of the parameters and for some initial conditions.

The boundedness characterization (U, U), next to a special case of System (2.28), means that there exist solutions in which the first component of the solution in this special case of the system is unbounded in some range of the parameters and for some initial conditions and also there exist solutions in which the second component is unbounded in some range of the parameters and for some initial conditions.

$$(4,1): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \alpha_2 \quad (\mathrm{U},\mathrm{B})$$

$$(4,2): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2}{y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(4,3): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(4,4): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \gamma_2 y_n \quad (\mathrm{U},\mathrm{U})$$

$$(4,5): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \beta_2 \quad (\mathrm{U},\mathrm{B})$$

$$(4,6): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(4,7): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \beta_2 x_n \quad (\mathrm{U},\mathrm{U})$$

$$(4,8): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{x_n}{y_n} \quad (\mathrm{U},\mathrm{U})$$

$$(4,9): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \gamma_2 \quad (\mathrm{U},\mathrm{B})$$

$$(4,10): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2}{1+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(4,11): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2}{1+x_n} \quad (\mathrm{U},\mathrm{B})$$

$$(4,12): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(4,13): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\gamma_2 y_n}{1+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(4,14): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\gamma_2 y_n}{1+x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(4,15): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(4,16): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\beta_2 x_n}{1+y_n} \quad (\mathrm{U},\mathrm{U})$$

$$(4,17):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\beta_2 x_n}{1+x_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,18):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\beta_2 x_n}{x_n+y_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,19):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\alpha_2+\gamma_2 y_n\quad (\mathrm{U},\mathrm{U})$$

$$(4,20):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{y_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,21):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{x_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,22):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\alpha_2+x_n\quad (\mathrm{U},\mathrm{U})$$

$$(4,23):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+x_n}{y_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,24):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+x_n}{x_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,25):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\beta_2 x_n+\gamma_2 y_n\quad (\mathrm{U},\mathrm{U})$$

$$(4,26):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{x_n+\gamma_2 y_n}{y_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,27):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,28):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{1+y_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,29):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{1+x_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,30):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{x_n+y_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,31):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+x_n}{A_2+y_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,32):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+x_n}{A_2+x_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,33):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{x_n+y_n}\quad (\mathrm{U},\mathrm{B})$$

$$(4,34):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{1+y_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,35):\quad x_{n+1}=\beta_1 x_n,\quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{1+x_n}\quad (\mathrm{U},\mathrm{U})$$

$$(4,36): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,37): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,38): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,39): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,40): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \alpha_2 + x_n + \gamma_2 y_n \quad (\mathrm{U}, \mathrm{U})$$

$$(4,41): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(4,42): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(4,43): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,44): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,45): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,46): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(4,47): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(4,48): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(4,49): \quad x_{n+1} = \beta_1 x_n, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,1): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \alpha_2 \quad (\mathrm{U}, \mathrm{B})$$

$$(6,2): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2}{y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,3): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(6,4): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \gamma_2 y_n \quad (\mathrm{U}, \mathrm{U})$$

$$(6,5): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \beta_2 \quad (\mathrm{U}, \mathrm{B})$$

$$(6,6): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(6,7): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \beta_2 x_n \quad (\mathrm{B}, \mathrm{B})$$

$$(6,8): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{x_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(6,9): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \gamma_2 \quad (\mathrm{U}, \mathrm{B})$$

$$(6,10): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2 + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,11): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2}{1 + x_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,12): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,13): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,14): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{1 + x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(6,15): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,16): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{1 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(6,17): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{1 + x_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,18): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,19): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \alpha_2 + \gamma_2 y_n \quad (\mathrm{U}, \mathrm{U})$$

$$(6,20): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(6,21): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(6,22): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \alpha_2 + x_n \quad (\mathrm{B}, \mathrm{B})$$

$$(6,23): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(6,24): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(6,25): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \beta_2 x_n + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U}) \text{ See Conjecture 1.1.}$$

$$(6,26): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(6,27): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(6,28): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,29): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(6,30): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,31): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(6,32): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + x_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,33): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,34): \quad x_{n+1} = \frac{x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(6,35): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(6,36): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,37): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,38): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,39): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,40): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \alpha_2 + x_n + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(6,41): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(6,42): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(6,43): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,44): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,45): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,46): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(6,47): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(6,48): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(6,49): \quad x_{n+1} = \frac{\beta_1 x_n}{y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + x_n + y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,1): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \alpha_2 \quad (\mathrm{U},\mathrm{B})$$

$$(14,2): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \frac{\alpha_2}{y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,3): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,4): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \gamma_2 y_n \quad (\mathrm{U},\mathrm{U})$$

$$(14,5): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \beta_2 \quad (\mathrm{U},\mathrm{B})$$

$$(14,6): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,7): \quad x_{n+1} = \frac{\beta_1 x_n}{1 + y_n}, \quad y_{n+1} = \beta_2 x_n \quad (\mathrm{B},\mathrm{B})$$

$$(14,8): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{x_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,9): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \gamma_2 \quad (\mathrm{U},\mathrm{B})$$

$$(14,10): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,11): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2}{1+x_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,12): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,13): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,14): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{1+x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,15): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,16): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2+y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,17): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{1+x_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,18): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,19): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \alpha_2 + \gamma_2 y_n \quad (\mathrm{U},\mathrm{U})$$

$$(14,20): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2+\gamma_2 y_n}{y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,21): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2+\gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,22): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \alpha_2 + x_n \quad (\mathrm{B},\mathrm{B})$$

$$(14,23): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2+\beta_2 x_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,24): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2+\beta_2 x_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,25): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \beta_2 x_n + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(14,26): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{x_n+\gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,27): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,28): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{A_2+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,29): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{1+x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,30): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,31): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{A_2+y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,32): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{A_2+x_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,33): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,34): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{A_2+y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,35): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{1+x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,36): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,37): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2}{A_2+B_2 x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,38): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\gamma_2 y_n}{A_2+B_2 x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,39): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\beta_2 x_n}{A_2+B_2 x_n+y_n} \quad (\mathrm{U},\mathrm{B})$$

$$(14,40): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\alpha_2+x_n+\gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(14,41): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(14,42): \quad x_{n+1}=\frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{x_n} \quad (\mathrm{U},\mathrm{U})$$

$$(14,43): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(14,44): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(14,45): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(14,46): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(14,47): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + x_n} \quad (\mathrm{U}, \mathrm{U})$$

$$(14,48): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

$$(14,49): \quad x_{n+1} = \frac{\beta_1 x_n}{1+y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{U}, \mathrm{B})$$

Appendix 2

The boundedness character of the rational system:

$$\left.\begin{aligned} x_{n+1} &= \frac{\beta_1 x_n}{A_1 + x_n + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots . \tag{2.29}$$

The boundedness characterization (B, B), next to a special case of System (2.29), means that both components of every solution of the system, in this special case, are bounded.

The boundedness characterization (B, U), next to a special case of System (2.29), means that the first component of every solution in this special case of the system is always bounded and there exist solutions in which the second component is unbounded in some range of the parameters and for some initial conditions.

The boundedness characterization (U, B), next to a special case of System (2.29), means that the second component of every solution in this special case of the system is always bounded and there exist solutions in which the first component is unbounded in some range of the parameters and for some initial conditions.

The boundedness characterization (U, U), next to a special case of System (2.29), means that there exist solutions in which the first component of the solution in this special case of the system is unbounded in some range of the parameters and for some initial conditions and also there exist solutions in which the second component is unbounded in some range of the parameters and for some initial conditions.

$$(5,1): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \alpha_2 \quad (\mathrm{B},\mathrm{B})$$

$$(5,2): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,3): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2}{x_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,4): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(5,5): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \beta_2 \quad (\mathrm{B},\mathrm{B})$$

$$(5,6): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(5,7): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \beta_2 x_n \quad (\mathrm{B},\mathrm{B})$$

$$(5,8): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{x_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,9): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \gamma_2 \quad (\mathrm{B},\mathrm{B})$$

$$(5,10): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,11): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2}{1 + x_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,12): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,13): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,14): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\gamma_2 y_n}{1 + x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(5,15): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,16): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,17): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n}{1 + x_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,18): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,19): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \alpha_2 + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(5,20): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(5,21): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(5,22): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \alpha_2 + x_n \quad (\mathrm{B}, \mathrm{B})$$

$$(5,23): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,24): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,25): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \beta_2 x_n + \gamma_2 y_n \quad (\mathrm{B}, \mathrm{U})$$

$$(5,26): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,27): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(5,28): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,29): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(5,30): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,31): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,32): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + x_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,33): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,34): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,35): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(5,36): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,37): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,38): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,39): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,40): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \alpha_2 + x_n + \gamma_2 y_n \quad (\mathrm{B}, \mathrm{U})$$

$$(5,41): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,42): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(5,43): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,44): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,45): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,46): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,47): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(5,48): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(5,49): \quad x_{n+1} = \beta_1, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,1): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \alpha_2 \quad (\mathrm{B}, \mathrm{B})$$

$$(13,2): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,3): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(13,4): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \gamma_2 y_n \quad (\mathrm{B}, \mathrm{U})$$

$$(13,5): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \beta_2 \quad (\mathrm{B}, \mathrm{B})$$

$$(13,6): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(13,7): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \beta_2 x_n \quad (\mathrm{B}, \mathrm{B})$$

$$(13,8):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{x_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,9):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\gamma_2\quad (\mathrm{B},\mathrm{B})$$

$$(13,10):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,11):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2}{1+x_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,12):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,13):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\gamma_2y_n}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,14):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\gamma_2y_n}{1+x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(13,15):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\gamma_2y_n}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,16):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\beta_2x_n}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,17):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\beta_2x_n}{1+x_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,18):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\beta_2x_n}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,19):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\alpha_2+\gamma_2y_n\quad (\mathrm{B},\mathrm{U})$$

$$(13,20):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2+\gamma_2y_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,21):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2+\gamma_2y_n}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(13,22):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\alpha_2+x_n\quad (\mathrm{B},\mathrm{B})$$

$$(13,23):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2x_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(13,24):\quad x_{n+1}=\frac{\beta_1x_n}{A_1+x_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2x_n}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(13,25): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \beta_2 x_n + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(13,26): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,27): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(13,28): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,29): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(13,30): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,31): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,32): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + x_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,33): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,34): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,35): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(13,36): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,37): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,38): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,39): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,40): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \alpha_2 + x_n + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(13,41): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(13,42): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(13,43): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,44): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,45): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,46): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,47): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(13,48): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(13,49): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,1): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \alpha_2 \quad (\mathrm{B}, \mathrm{B})$$

$$(15,2): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,3): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(15,4): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \gamma_2 y_n \quad (\mathrm{B}, \mathrm{U})$$

$$(15,5): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \beta_2 \quad (\mathrm{B}, \mathrm{B})$$

$$(15,6): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(15,7): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \beta_2 x_n \quad (\mathrm{B}, \mathrm{B})$$

$$(15,8): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{x_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,9): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \gamma_2 \quad (\mathrm{B}, \mathrm{B})$$

$$(15,10): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,11): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{1 + x_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,12): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,13): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,14): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{1 + x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(15,15): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,16): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,17): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{1 + x_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,18): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{x_n + y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,19): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \alpha_2 + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(15,20): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,21): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(15,22): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \alpha_2 + x_n \quad (\mathrm{B},\mathrm{B})$$

$$(15,23): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,24): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n} \quad (\mathrm{B},\mathrm{U})$$

$$(15,25): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \beta_2 x_n + \gamma_2 y_n \quad (\mathrm{B},\mathrm{U})$$

$$(15,26): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B},\mathrm{B})$$

$$(15,27): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(15,28): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,29): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(15,30): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,31): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,32): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + x_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,33): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,34): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,35): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(15,36): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,37): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,38): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,39): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,40): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \alpha_2 + x_n + \gamma_2 y_n \quad (\mathrm{B}, \mathrm{U})$$

$$(15,41): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,42): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(15,43): \quad x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(15,44):\quad x_{n+1}=\frac{\beta_1 x_n}{B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+x_n}{A_2+B_2x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(15,45):\quad x_{n+1}=\frac{\beta_1 x_n}{B_1x_n+y_n},\quad y_{n+1}=\frac{\beta_2x_n+\gamma_2y_n}{A_2+B_2x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(15,46):\quad x_{n+1}=\frac{\beta_1 x_n}{B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2x_n+\gamma_2y_n}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(15,47):\quad x_{n+1}=\frac{\beta_1 x_n}{B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2x_n+\gamma_2y_n}{A_2+x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(15,48):\quad x_{n+1}=\frac{\beta_1 x_n}{B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2x_n+\gamma_2y_n}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(15,49):\quad x_{n+1}=\frac{\beta_1 x_n}{B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+x_n+\gamma_2y_n}{A_2+B_2x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,1):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\alpha_2\quad (\mathrm{B},\mathrm{B})$$

$$(38,2):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,3):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(38,4):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\gamma_2y_n\quad (\mathrm{B},\mathrm{U})$$

$$(38,5):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\beta_2\quad (\mathrm{B},\mathrm{B})$$

$$(38,6):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\gamma_2y_n}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(38,7):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\beta_2x_n\quad (\mathrm{B},\mathrm{B})$$

$$(38,8):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{x_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,9):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\gamma_2\quad (\mathrm{B},\mathrm{B})$$

$$(38,10):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,11):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2}{1+x_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,12):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,13):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\gamma_2 y_n}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,14):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\gamma_2 y_n}{1+x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(38,15):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\gamma_2 y_n}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,16):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\beta_2 x_n}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,17):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\beta_2 x_n}{1+x_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,18):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\beta_2 x_n}{x_n+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,19):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\alpha_2+\gamma_2 y_n\quad (\mathrm{B},\mathrm{U})$$

$$(38,20):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,21):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(38,22):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\alpha_2+x_n\quad (\mathrm{B},\mathrm{B})$$

$$(38,23):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,24):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2 x_n}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(38,25):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\beta_2 x_n+\gamma_2 y_n\quad (\mathrm{B},\mathrm{U})$$

$$(38,26):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{x_n+\gamma_2 y_n}{y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,27):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{x_n}\quad (\mathrm{B},\mathrm{U})$$

$$(38,28):\quad x_{n+1}=\frac{\beta_1 x_n}{A_1+B_1x_n+y_n},\quad y_{n+1}=\frac{\alpha_2+\gamma_2 y_n}{A_2+y_n}\quad (\mathrm{B},\mathrm{B})$$

$$(38,29): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(38,30): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,31): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,32): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{A_2 + x_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,33): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,34): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,35): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{1 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(38,36): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,37): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,38): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,39): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,40): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \alpha_2 + x_n + \gamma_2 y_n \quad (\mathrm{B}, \mathrm{U})$$

$$(38,41): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,42): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(38,43): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,44): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,45): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,46): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,47): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + x_n} \quad (\mathrm{B}, \mathrm{U})$$

$$(38,48): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$

$$(38,49): \quad x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n} \quad (\mathrm{B}, \mathrm{B})$$