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On the period five trichotomy of all positive solutions of $x_{n+1} = (p + x_{n-2})/x_n$

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Abstract

We study the behavior of all positive solutions of the difference equation in the title, where p is a positive real parameter and the initial conditions x_{-2}, x_{-1}, x_0 are positive real numbers. For all the values of the positive parameter p there exists a unique positive equilibrium $\bar{x}$ which satisfies the equation

$$\bar{x}^2 = \bar{x} + p.$$

We show that if $0 < p < 1$ or $p \geqslant 2$ every positive bounded solution of the equation in the title converges to the positive equilibrium $\bar{x}$. When $0 < p < 1$ we show the existence of unbounded solutions. When $p \geqslant 2$ we show that the positive equilibrium is globally asymptotically stable. Finally we conjecture that when $1 < p < 2$, the positive equilibrium is globally asymptotically stable.

Keywords: Difference equation; Boundedness; Global stability; Periodic solution

1. Introduction and preliminaries

Consider the difference equation

$$x_{n+1} = \frac{p + x_{n-2}}{x_n}, \quad n = 0, 1, \ldots, \tag{1}$$

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where p is a positive real parameter, and the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers.

The following theorem, which is a direct consequence of the conditions given in [4], gives explicit conditions for the local asymptotic stability of the positive equilibrium $\bar{x}$ of Eq. (1).

Theorem A. *The positive equilibrium $\bar{x}$ of Eq.* (1) *is locally asymptotically stable when $p > 1$and unstable when* $0 < p < 1$.

When $p = 1$ local stability analysis fails. In this case a period five cycle appears. It has been conjectured in [1] that Eq. (1) possesses the following period-five trichotomy:

(a) Every solution of Eq. (1) has a finite limit if and only if $p > 1$.
(b) Every solution of Eq. (1) converges to a period-five solution if and only if $p = 1$.
(c) Eq. (1) has positive unbounded solutions if and only if $0 < p < 1$.

Part (b) of the conjecture has been verified in [1]. See also [2,3].

2. Global analysis of positive solutions of Eq. (1)

Theorem 1. *Let $\{x_n\}_{n=-2}^{\infty}$ be a positive solution of Eq.* (1) *for which there exists $N \geqslant -2$ such that $x_N < \bar{x}$ and $x_{N+1} \geqslant \bar{x}$, or $x_N \geqslant \bar{x}$ and $x_{N+1} < \bar{x}$. Then the solution $\{x_n\}_{n=-2}^{\infty}$ oscillates about the equilibrium $\bar{x}$ with every semicycle* (*except possibly the first*) *having at most two terms.*

Proof. Let $N \geqslant -2$ such that $x_N < \bar{x} \leqslant x_{N+1}$. The case where $x_{N+1} < \bar{x} \leqslant x_N$ is similar and will be omitted. Now suppose that the positive semicycle beginning with the term x_{N+1} has two terms. Then $x_N < \bar{x} \leqslant x_{N+2}$ and so

$$x_{N+3} = \frac{p + x_N}{x_{N+2}} < \frac{p + \bar{x}}{\bar{x}} = \bar{x}.$$

The proof is complete. □

In view of Theorem 1 and without loss of generality, when we refer to an oscillatory solution of Eq. (1), we will assume that the first semicycle of that solution, positive or negative, will contain at most two terms.

Theorem 2. *All nonoscillatory solutions of Eq.* (1) *converge to the positive equilibrium $\bar{x}$.*

Proof. We will give the proof of the theorem in the case of a single positive semicycle. The case of a single negative semicycle is similar and will be omitted. Assume that $x_n \geqslant \bar{x}$ for all $n \geqslant -2$. We first claim that for this solution

$$x_{n-2} \geqslant x_n \quad \text{for all } n = 0, 1, \ldots.$$

For the sake of contradiction assume that there exists $N \geqslant 0$ such that $x_{N-2} < x_N$. Using Eq. (1) we have

$$x_{N+1} = \frac{p + x_{N-2}}{x_N} < \frac{p + x_N}{x_N} \leqslant \bar{x},$$

which is a contradiction and so

$$\bar{x} \leqslant x_n \leqslant x_{n-2} \quad \text{for } n = 0, 1, \ldots.$$

In addition for $i = 0, 1$ there exists α_i such that

$$\lim_{n\to\infty} x_{2n+i} = \alpha_i.$$

It follows that $\{\alpha_0, \alpha_1, \alpha_0, \alpha_1, \ldots\}$ is a periodic solution of not necessarily prime period two. On the other hand Eq. (1) has no prime period two solutions, and so $\alpha_0 = \alpha_1 = \bar{x}$. The proof is complete. $\square$

The following will be useful in the sequel. Set

$$f(x, y) = p^3 x + p^3 + (p^2 - p)y,$$
$$g(x, y) = (p^3 - p^2)x + (p^2 - p)xy + (p^2 - p)y + (p - 1)y^2,$$

and

$$h(x, y) = p^3 x + p^3 + (p^2 - p - \bar{x})y.$$

It holds that

$$h(\bar{x}, \bar{x}) = g(\bar{x}, \bar{x}). \tag{2}$$

The following lemma, the proof of which follows by a simple computation and will be omitted, provides three identities which will be useful in our study.

Lemma 1. *Every positive solution of Eq.* (1) *satisfies the following identities*:

$$x_{n+4} - x_{n-1} = \frac{p x_{n+3} - (p-1)x_{n+1} - x_{n-2}}{x_{n+3}}, \quad n = 0, 1, \ldots, \tag{3}$$

$$x_{n+6} - x_{n+1} = \frac{p x_{n+4} - (p-1)x_{n+1} - x_{n-1}}{p + x_{n+2}}, \quad n = 0, 1, \ldots, \tag{4}$$

$$x_{n+10} - x_{n+5} = \frac{f(x_{n+4}, x_{n+1}) - g(x_{n+5}, x_{n+2}) - x_{n+2}x_{n+1}}{(p + x_{n+6})(p + x_{n+4})(p + x_{n+2})}, \quad n = 0, 1, \ldots. \tag{5}$$

Lemma 2. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a positive oscillatory solution of Eq.* (1), *which is bounded from above and below. Let* l_0, l_1, *and* l_2 *be the limits of three consecutive subsequences* x_{n_i}, x_{n_i+1}, *and* x_{n_i+2} *of* $\{x_n\}_{n=-2}^{\infty}$. *These limits cannot be all less than* $\bar{x}$. *In addition they cannot be all greater than or equal to* $\bar{x}$, *unless they are all equal to* $\bar{x}$.

Proof. To prove this lemma we will consider several cases.

Case 1: $l_0, l_1, l_2 \neq \bar{x}$. Let $0 < \epsilon < \min(|l_0 - \bar{x}|, |l_1 - \bar{x}|, |l_2 - \bar{x}|)$. Then there exists $N \geqslant -2$ such that

$$l_0 - \epsilon < x_{n_N} < l_0 + \epsilon, \qquad l_1 - \epsilon < x_{n_N+1} < l_1 + \epsilon, \qquad l_2 - \epsilon < x_{n_N+2} < l_2 + \epsilon.$$

Suppose for the sake of contradiction that either $\max(l_0, l_1, l_2) < \bar{x}$, or $\min(l_0, l_1, l_2) \geqslant \bar{x}$. It follows that either

$$x_{n_N} < \bar{x}, \qquad x_{n_N+1} < \bar{x}, \qquad x_{n_N+2} < \bar{x},$$

or

$$x_{n_N} > \bar{x}, \qquad x_{n_N+1} > \bar{x}, \qquad x_{n_N+2} > \bar{x},$$

which in view of Theorem 1 yields a contradiction.

Case 2: $l_0 = \bar{x}$. There exist subsequences of the solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1), namely $\{x_{n_i+k}\}_{i=1}^{\infty}$, where $k = -5, -4, -3, -2, -1, 0, 1, 2$ such that

$$\lim_{i\to\infty} x_{n_i+k} = l_k.$$

We will show that $l_1 < \bar{x}$ or $l_2 < \bar{x}$ or $l_1 = l_2 = \bar{x}$. Suppose for the sake of contradiction that $l_1 \geqslant \bar{x}$ and $l_2 \geqslant \bar{x}$ and also $l_1 \neq \bar{x}$ or $l_2 \neq \bar{x}$. Using Eq. (1) we have

$$l_i = \frac{p + l_{i-3}}{l_{i-1}}, \quad i = -2, -1, 0, 1, 2,$$

from which it follows that either

$$l_1 = l_2 = \bar{x} \tag{6}$$

or

$$l_{i-3} > \bar{x} \quad \text{for } i = 0, 1, 2 \quad \text{or} \quad l_{i-3} > \bar{x} \quad \text{for } i = -2, -1, 0. \tag{7}$$

If (6) holds we have a contradiction. On the other hand if (7) holds arguing as in Case 1, we get a contradiction. The cases where $l_1 = \bar{x}$, $l_2 = \bar{x}$ are similar and will be omitted. The proof is complete. □

Lemma 3. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a positive oscillatory solution of Eq.* (1) *such that*

$$x_k \geqslant \max x_i, \quad i = k - 10, k - 9, \ldots, k - 1, \tag{8}$$

with $k \geqslant 10$. *Then the following are true*:

$$x_k \geqslant \bar{x}, \tag{9}$$

$$x_{k+j} > x_{k+j-5}, \quad j = -7, -6, \ldots, 2, \tag{10}$$

and

$$x_{k-9} \leqslant x_{k-4} \leqslant x_{k+1} \leqslant \bar{x}. \tag{11}$$

Proof. Since $\{x_n\}_{n=-2}^{\infty}$ oscillates about $\bar{x}$, in view of (8) and Theorem 1, (9) follows. Furthermore from (8) we have

$$px_k \geqslant (p-1)x_{k-2} + x_{k-5} \quad \text{and} \quad px_k \geqslant (p-1)x_{k-3} + x_{k-5}. \tag{12}$$

In view of (3) and (4), we get

$$x_{k+1} \geqslant x_{k-4} \quad \text{and} \quad x_{k+2} \geqslant x_{k-3}. \tag{13}$$

From Eq. (1), we have

$$x_{k+1} = \frac{p + x_{k-1}}{x_{k+2}},$$

and since $x_{k+2} \geqslant x_{k-3}$, $x_{k+1} \geqslant x_{k-4}$, it follows that

$$x_{k-1} \geqslant x_{k-6}.$$

Similarly we can show that

$$x_{k+j} \geqslant x_{k+j-5}, \quad j = -7, -6, \ldots, -2.$$

Using Eq. (1), and in view of (8) and (9), we have

$$x_{k+1} = \frac{p + x_{k-2}}{x_k} \leqslant \frac{p + x_k}{x_k} \leqslant \bar{x}. \tag{14}$$

Therefore in view of (10) and (14), we have

$$x_{k-9} \leqslant x_{k-4} \leqslant x_{k+1} \leqslant \bar{x}.$$

The proof is complete. □

Lemma 4. *Let $\{x_n\}_{n=-2}^{\infty}$ be a positive solution of Eq.* (1), *which is bounded from above and below. Let*

$$s = \limsup_{n\to\infty} x_n \quad \textit{and} \quad i = \liminf_{n\to\infty} x_n.$$

If $s = \bar{x}$, then

$$i = \bar{x}.$$

Proof. Assume $s = \bar{x}$. There exist subsequences $\{x_{n_i+k}\}_{i=1}^{\infty}$, $k = -3, -2, \ldots,$ of the solution $\{x_n\}_{n=-2}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i} = i_0 = i \leqslant i_k = \lim_{i\to\infty} x_{n_i+k} \leqslant \bar{x}.$$

In addition $\{i_k\}_{k=-3}^{\infty}$ is a solution of Eq. (1) and so

$$i_0 = \frac{p + i_{-3}}{i_{-1}} \geqslant \frac{p + i_0}{\bar{x}},$$

which implies

$$\bar{x} \geqslant \frac{p + i_0}{i_0} \geqslant \bar{x}.$$

Hence, $i_0 = \bar{x}$. The proof is complete. □

Lemma 5. *Assume $p \geqslant 2$. Let $\{x_n\}_{n=-2}^{\infty}$ be a positive nontrivial solution of Eq.* (1). *If for some $N \geqslant -2$ we have*

$$x_{N+4} \leqslant \bar{x}, \quad x_{N+1} \leqslant \bar{x}, \quad x_{N+5} \geqslant \bar{x}, \quad x_{N+2} \geqslant \bar{x}, \tag{15}$$

then

$$x_{N+10} < x_{N+5}. \tag{16}$$

Proof. Suppose for the sake of contradiction that

$$x_{N+10} \geqslant x_{N+5}. \tag{17}$$

In view of (5) and (17) we have

$$f(x_{N+4}, x_{N+1}) \geqslant g(x_{N+5}, x_{N+2}) + x_{N+2}x_{N+1}. \tag{18}$$

When $p \geqslant 2$, it holds that $p^2 - p - \bar{x} \geqslant 0$. In view of (15), (18) implies that

$$h(x_{N+4}, x_{N+1}) \geqslant g(x_{N+5}, x_{N+2}),$$

and so

$$h(\bar{x}, \bar{x}) > g(\bar{x}, \bar{x})$$

which in view of (2) is a contradiction. The proof is complete. □

3. Boundedness and convergence of positive solutions of Eq. (1) in the case $p \geqslant 2$

Theorem 3. *Let $p \geqslant 2$, and $\{x_n\}_{n=-2}^{\infty}$ be an oscillatory nontrivial positive solution of Eq.* (1). *Then $\{x_n\}_{n=-2}^{\infty}$ is bounded from above and below. In addition we have*

$$x_n \leqslant \max_{0\leqslant i\leqslant 9} x_i \tag{19}$$

for all $n \geqslant 10$.

Proof. We first show that

$$x_k \leqslant \max_{k-10\leqslant i\leqslant k-1} x_i \tag{20}$$

for every $k \geqslant 10$. Suppose for the sake of contradiction that there exists $k \geqslant 10$ such that

$$x_k > \max_{k-10\leqslant i\leqslant k-1} x_i. \tag{21}$$

In view of Lemma 3, (21) implies that

$$x_k > \bar{x}, \tag{22}$$

$$x_{k+j} \geqslant x_{k+j-5}, \quad j = -7, -6, \ldots, 2, \tag{23}$$

and

$$x_{k-9} \leqslant x_{k-4} \leqslant x_{k+1} < \bar{x}. \tag{24}$$

Case 1. If $x_{k-10} < \bar{x}$, in view of (22)–(24), it holds that

$$\max_{9\leqslant i\leqslant 11} x_{k-i} < \bar{x} \quad \text{or} \quad \max_{10\leqslant i\leqslant 12} x_{k-i} < \bar{x}$$

or

$$x_{k+1} = \frac{p + x_{k-2}}{x_k} < \frac{p + x_{k-6}}{x_{k-3}} = x_{k-4},$$

which in view of (10) and Theorem 1 yields a contradiction.

Case 2. If $x_{k-10} \geqslant \bar{x}$ with the use of (23) and (24) it follows that there exists $j = 2, 3, 4$ such that $\max\{x_{k-2j-3}, x_{k-2j}\} \leqslant \bar{x} \leqslant \min\{x_{k-2j-2}, x_{k-2j+1}\}$ and so in view of Lemma 5 we have $x_{k-2j+6} < x_{k-2j+1}$ which is a contradiction and proves (20).

Using (20) we show that (19) holds for $k = 10, 11, \ldots, 20$. Let $N > 20$ be the smallest integer such that (19) does not hold. Then

$$x_N > \max_{0\leqslant i\leqslant 9} x_i \quad \text{and} \quad x_{N-j} \leqslant \max_{0\leqslant i\leqslant 9} x_i, \quad j = 1, 2, \ldots, 10. \tag{25}$$

Combining (20) and (25) we get

$$\max_{0\leqslant i\leqslant 9} x_i < x_N \leqslant \max_{N-10\leqslant i\leqslant N-1} x_i \leqslant \max_{0\leqslant i\leqslant 9} x_i,$$

which is a contradiction and so (19) holds true for all $n \geqslant 10$. Finally, if $\{x_n\}_{n=-2}^{\infty}$ is not bounded from below there exists a subsequence which converges to zero. But then using Eq. (1), we can easily see that the next subsequence goes to infinity which is a contradiction. The proof is complete. □

Theorem 4. *Let $p \geqslant 2$. Then every positive solution $\{x_n\}_{n=-2}^{\infty}$ of Eq.* (1) *converges to the unique positive equilibrium of Eq.* (1).

Proof. In the case where the solution is nonoscillatory the proof follows from Theorem 2. Therefore we assume that the positive solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (1) oscillates about $\bar{x}$. First we show that

$$\limsup_{n\to\infty} x_n = s = \bar{x}.$$

In view of Theorem 3, $\{x_n\}_{n=-2}^{\infty}$ is bounded from above and below. There exist subsequences $\{x_{n_i+k}\}_{i=1}^{\infty}$, $k = -2, -1, \ldots,$ such that

$$\lim_{i\to\infty} x_{n_i+10} = l_{10} = s \geqslant l_k = \lim_{i\to\infty} x_{n_i+k} \quad \text{and} \quad s = l_{10} \geqslant \bar{x}, \tag{26}$$

and the sequence $\{l_k\}_{k=-2}^{\infty}$ is a solution of Eq. (1). In view of (8), (10), and (11), (26) implies

$$l_{i+5} \geqslant l_i, \quad i = -2, -1, \ldots, 7, \quad \text{and} \quad l_1 \leqslant l_6 \leqslant l_{11} \leqslant \bar{x}. \tag{27}$$

We now claim that $l_9 \geqslant \bar{x}$. For the sake of contradiction we consider the following three cases.

Case 1. If $l_9 < \bar{x}$ and $l_0 < \bar{x}$, from (27) we have $l_{-1} \leqslant l_4 \leqslant l_9 < \bar{x}$ and $l_1 \leqslant l_6 \leqslant l_{11} \leqslant \bar{x}$. Since $\max\{l_{-1}, l_0\} < \bar{x}$, in view of Lemma 2 it follows that $l_1 = l_{11} = l_6 = \bar{x}$. Using Eq. (1) we get

$$\bar{x} = l_{11} = \frac{p + l_8}{l_{10}} \leqslant \frac{p + l_{10}}{l_{10}} \leqslant \bar{x},$$

which implies that $l_9 = (p + l_6)/l_8 = \bar{x}$, a contradiction.

Case 2. If $l_9 < \bar{x}$, $l_0 \geqslant \bar{x}$, and $l_2 < \bar{x}$, from (27) we have $l_{-1} \leqslant l_4 < \bar{x}$ and in addition in view of Lemma 2, it follows that $l_3 \geqslant \bar{x}$. Hence, in view of Lemma 5 we conclude $l_8 < l_3$ which is a contradiction.

Case 3. If $l_9 < \bar{x}$, $l_0 \geqslant \bar{x}$, and $l_2 \geqslant \bar{x}$, from (27) we have $\bar{x} \leqslant l_0 \leqslant l_5$, $l_4 \leqslant l_9 < \bar{x}$, and $l_1 \leqslant \bar{x}$. In view of Lemma 5 it follows that $l_{10} < l_5$ which is a contradiction and the proof of the claim is complete. Hence

$$l_9 \geqslant \bar{x}.$$

Since $l_{10} \geqslant l_{12}$ with the use of Eq. (1) we get

$$l_8 \geqslant l_9 \geqslant \bar{x}.$$

In view of Lemma 2 we have

$$l_{10} = l_8 = l_9 = \bar{x}.$$

In view of Lemma 4 the proof is complete. □

Theorem 4 shows that when $p \geqslant 2$, $\bar{x}$ is a global attractor of all positive solutions of Eq. (1). From Theorem A, we have that when $p \geqslant 2$, $\bar{x}$ is locally asymptotically stable, and so when $p \geqslant 2$, $\bar{x}$ is globally asymptotically stable.

4. Unbounded solutions of Eq. (1) in the case $0 < p < 1$

Lemma 6. *Let* $0 < p < 1$, *and let* $\{x_n\}_{n=-2}^{\infty}$ *be a positive solution of Eq.* (1) *for which there exists* $N \geqslant 3$ *such that*

$$x_N \geqslant x_n, \quad n \geqslant -2. \tag{28}$$

Then

$$x_n = \bar{x}, \quad n \geqslant -2.$$

Proof. Using (28) with $n = N + 5$, (4) implies that

$$px_{N+3} + (1-p)x_N \leqslant x_{N-2}. \tag{29}$$

Furthermore, in view of (28), we have $x_N \geqslant x_{N+3}$ and so (29) implies that

$$x_{N+3} \leqslant x_{N-2}. \tag{30}$$

In view of (3), we have

$$x_{N+3} - x_{N-2} = \frac{x_{N+2} - x_{N-3}}{x_{N+2}} + \frac{(p-1)(x_{N+2} - x_N)}{x_{N+2}}. \tag{31}$$

From (28), we have $x_N \geqslant x_{N+2}$. Therefore in view of (30), (31) implies that

$$x_{N+2} \leqslant x_{N-3}. \tag{32}$$

In addition, from (28) we have

$$x_N \geqslant x_{N-5}, \tag{33}$$

and by using Eq. (1) we get

$$x_{N+3} = \frac{p + x_N}{x_{N+2}} \geqslant \frac{p + x_{N-5}}{x_{N-3}} = x_{N-2}. \tag{34}$$

From (30) and (34) we have $x_{N+3} = x_{N-2}$, and so (28) and (29) imply that

$$x_N = x_{N-2}.$$

In addition

$$px_{N+3} + (1-p)x_N = px_{N+3} + (1-p)x_{N-2} = x_{N-2},$$

from which it follows with the use of (4) that $x_{N+5} = x_N$. It is also true that

$$px_N + (1-p)x_{N-2} = x_N \geqslant x_{N-5}, \tag{35}$$

from which it follows with the use of (3) that $x_{N+1} \geqslant x_{N-4}$. Using Eq. (1) we get

$$x_{N+4} = \frac{p + x_{N+1}}{x_{N+3}} \geqslant \frac{p + x_{N-4}}{x_{N-2}} = x_{N-1}. \tag{36}$$

In view of (3) we have

$$x_{N+4} - x_{N-1} = \frac{x_{N+3} - x_{N-2}}{x_{N+3}} + \frac{(p-1)(x_{N+3} - x_{N+1})}{x_{N+3}}. \tag{37}$$

Since $x_{N+3} = x_{N-2}$, (36) and (37) imply that

$$x_{N+3} \leqslant x_{N+1}. \tag{38}$$

From Eq. (1) with the use of (38) we get

$$x_N \leqslant x_{N-1},$$

from which it follows, with the use of (28), that

$$x_N = x_{N-1}.$$

Therefore $x_{N+3} = x_N = x_{N-1} = x_{N-2}$. Using Eq. (1) we have

$$x_n = \bar{x}, \quad n \geqslant -2.$$

The proof is complete. $\square$

Theorem 5. *Let* $0 < p < 1$. *Then every positive solution of Eq.* (1) *is either unbounded or converges to the positive equilibrium of Eq.* (1).

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a positive solution of Eq. (1) which is bounded from above and below. Set

$$s = \limsup_{n\to\infty} x_n.$$

There exists subsequences of $\{x_n\}_{n=-2}^{\infty}$, namely $\{x_{n_i+k}\}_{i=1}^{\infty}$, $k = -2, -1, \ldots,$ such that

$$\lim_{i\to\infty} x_{n_i+4} = l_4 = s \geqslant \lim_{i\to\infty} x_{n_i+k} = l_k. \tag{39}$$

In addition the sequence $\{l_k\}_{k=-2}^{\infty}$ satisfies Eq. (1). In view of (39) and Lemma 6, we have

$$l_k = \bar{x} \quad \text{for all } k \geqslant -2,$$

and so

$$s = \bar{x}.$$

In view of Lemma 4, the proof is complete. $\square$

Lemma 7. *Let $0 < p < 1$ and $\{x_n\}_{n=-2}^{\infty}$ be a positive nontrivial oscillatory solution of Eq.* (1) *which is bounded from above and below. Then there exists $-2 \leqslant N \leqslant 2$ such that*

$$x_N = \sup\{x_n\}_{n=-2}^{\infty}. \tag{40}$$

Proof. If $N \geqslant 3$, in view of Lemma 6 we get a contradiction. On the other hand assume that

$$s = \sup\{x_n\}_{n=-2}^{\infty}$$

and

$$s > x_n \quad \text{for all } n \geqslant -2.$$

Since $\{x_n\}_{n=-2}^{\infty}$ oscillates about $\bar{x}$, we have $s > \bar{x}$. Furthermore there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=-2}^{\infty}$ such that

$$\lim_{i \to \infty} x_{n_i} = s > \bar{x},$$

which in view of Theorem 5 is a contradiction. □

Theorem 6. *Let $0 < p < 1$ and $\{x_n\}_{n=-2}^{\infty}$ be a positive nontrivial oscillatory solution of Eq.* (1) *such that*

$$\sup x_n \neq x_i, \quad i = -2, -1, 0, 1, 2. \tag{41}$$

Then $\{x_n\}_{n=-2}^{\infty}$ is an unbounded solution of Eq. (1).

Proof. Assume for the sake of contradiction that the solution $\{x_n\}_{n=-2}^{\infty}$ is bounded from above and below. In view of Lemma 7, we get a contradiction. The proof is complete. □

Conjecture. Let $1 < p < 2$. Then the positive equilibrium $\bar{x}$ of Eq. (1) is globally asymptotically stable.

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