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On the Dynamics of $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+x_n}$

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We investigate the existence of unbounded solutions and the period-two convergence of solutions of the equation in the title with the parameter γ positive, the remaining parameters nonnegative, and with nonnegative initial conditions.

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INTRODUCTION

We investigate the existence of unbounded solutions and the period-two convergence of solutions of the third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots \tag{1.1}$$

with the parameter γ positive, the remaining parameters α, β, δ and A nonnegative, and with nonnegative initial conditions such that the denominator is always positive.

The case $\delta = 0$, that is the second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + x_n}, \quad n = 0, 1, \ldots \tag{1.2}$$

was investigated in Refs. [8,9], where the following period-two trichotomy result was established for the solutions of Eq. (1.2). See also Ref. [11].

Theorem A [see refs. [8,9,11]). *The following period-two trichotomy result holds for Eq.* (1.2)*:*

(a) *Equation* (1.2) *has unbounded solutions if and only if*

$$\gamma > \beta + A.$$

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(b) *Every solution of Eq.* (1.2) *converges to a* (*not necessarily prime*) *period-two solution of Eq.* (1.2) *if and only if*

$$\gamma = \beta + A.$$

(c) *Every solution of Eq.* (1.2) *has a finite limit if and only if*

$$\gamma < \beta + A.$$

For Eq. (1.1) we were able to extend statements (a) and (b) of Theorem A as described by the following theorem which is the main result in the paper.

THEOREM 1

(a) *Assume that*

$$\gamma > \beta + A. \tag{1.3}$$

Then Eq. (1.1) *has unbounded solutions.*

(b) *Assume that*

$$\gamma = \beta + \delta + A \quad \text{and} \quad \beta + A > 0. \tag{1.4}$$

Then every solution of Eq. (1.1) *converges to a* (*not necessarily prime*) *period-two solution.*

Equation (1.1) does not have a trichotomy character in the spirit of Theorem A for Eq. (1.2). Actually it is not true that when

$$\gamma < \beta + \delta + A \tag{1.5}$$

every solution of Eq. (1.1) has a finite limit. This is true when $\delta = 0$, but when $\delta > 0$, Eq. (1.5) is not sufficient (even) for the local asymptotic stability of the equilibrium point of Eq. (1.1).

Some third order rational difference equations were investigated in Refs. [1–7,10]. The study of rational difference equations is quite challenging and rewarding. Third order rational equations with all of the variables x_n, x_{n-1} and x_{n-2} present in the equation are extremely difficult to handle and very little is known about them.

We believe that the methods and techniques which we develop to understand the dynamics of rational equations will also be useful in analyzing the equations in the mathematical models of various biological systems and other applications.

In the second section, we present the proof of statement (a) of Theorem 1, and in the third section we establish statement (b).

EXISTENCE OF UNBOUNDED SOLUTIONS

In this section we assume that

$$\gamma > \beta + \delta + A \tag{2.1}$$

and show that there exist solutions of Eq. (1.1) which are unbounded. Actually we exhibit a huge set of initial conditions through which the subsequences of even and odd terms of the solutions converge, one of them to ∞ and the other to

$$\frac{\beta\gamma + \delta A}{\gamma - \delta}$$

for all nonnegative values of the parameters α, β, γ, δ and A in the equation. Furthermore, our proof here extends and unifies all previously known results on the existence of unbounded solutions for all special cases of Eq. (1.1).

More precisely we establish the following result.

THEOREM 2 *Assume that Eq.* (2.1) *holds and let k be any number such that*

$$0 < k < \gamma - \beta - \delta - A.$$

Then every solution of Eq. (1.1) *with initial conditions* x_{-2}, x_{-1}, x_0 *such that*

$$x_{-2}, x_0 \in (0, \gamma - A) \quad \textit{and} \quad x_{-1} > \frac{\alpha + \gamma(\gamma - A)}{k}$$

is unbounded and more precisely

$$\lim_{n\to\infty} x_{2n+1} = \infty \quad \textit{and} \quad \lim_{n\to\infty} x_{2n} = \frac{\beta\gamma + \delta A}{\gamma - \delta}.$$

Proof Observe that

$$x_1 - x_{-1} = \frac{\alpha + \beta x_0 + \gamma x_{-1} + \delta x_{-2}}{A + x_0} - x_{-1} = \frac{\alpha + \beta x_0 + (\gamma - A - x_0)x_{-1} + \delta x_{-2}}{A + x_0}$$

and so

$$x_1 > x_{-1}.$$

Also,

$$\begin{aligned} x_2 - (\beta + \delta + k) &\leq \frac{\alpha + \beta x_1 + \gamma x_0 + \delta x_{-1}}{x_1} - (\beta + \delta + k) \\ &\leq \frac{\alpha + \gamma x_0 + \delta(x_{-1} - x_1) - k x_1}{x_1} \\ &< \frac{\alpha + \gamma(\gamma - A) - \alpha - \gamma(\gamma - A)}{x_1} = 0. \end{aligned}$$

Therefore,

$$x_2 < \beta + \delta + k < \gamma - A$$

and furthermore

$$x_3 = \frac{\alpha + \beta x_2 + \gamma x_1 + \delta x_0}{A + x_2} > \frac{\gamma}{\beta + \delta + A + k} x_1.$$

It follows by induction that for $n \geq 0$,

$$x_{2n} < \beta + \delta + k$$

and

$$x_{2n+1} > \frac{\gamma}{\beta + \delta + A + k} x_{2n-1}$$

and so, in particular,

$$\lim_{n\to\infty} x_{2n+1} = \infty. \tag{2.2}$$

Let S and I denote the following limits:

$$S = \limsup_{n\to\infty} x_{2n} \quad \text{and} \quad I = \liminf_{n\to\infty} x_{2n}.$$

Note that from Eq. (1.1),

$$x_{2n+1} = \frac{\alpha + \beta x_{2n} + \gamma x_{2n-1} + \delta x_{2n-2}}{A + x_{2n}} > \gamma \frac{x_{2n-1}}{A + x_{2n}}$$

and so for $n \geq 0$,

$$\frac{x_{2n-1}}{x_{2n+1}} < \frac{A + x_{2n}}{\gamma}.$$

Let $\epsilon > 0$ be given. Then clearly, in view of Eq. (2.2), there exists $N \geq 0$ such that

$$\frac{\alpha + \beta x_{2n+1} + \gamma x_{2n}}{A + x_{2n+1}} < \beta + \epsilon, \quad \text{for } n \geq N.$$

By using the above two inequalities, it follows from Eq. (1.1) that for $n \geq N$,

$$x_{2n+2} = \frac{\alpha + \beta x_{2n+1} + \gamma x_{2n} + \delta x_{2n-1}}{A + x_{2n+1}} < (\beta + \epsilon) + \frac{\delta}{\gamma}(A + x_{2n})$$
$$= \left(\beta + \frac{\delta A}{\gamma} + \epsilon\right) + \frac{\delta}{\gamma} x_{2n}.$$

By using the comparison principle and by taking limit superiors we find

$$S \leq \frac{\beta\gamma + \delta A + \gamma\epsilon}{\gamma - \delta}$$

and so clearly,

$$S \leq \frac{\beta\gamma + \delta A}{\gamma - \delta}.$$

When

$$\beta = A = 0,$$

we see that $S = 0$, that is $\lim_{n\to\infty} x_{2n} = 0$ and the proof is complete.

Next assume that

$$\beta + A > 0.$$

Clearly there exist subsequences $\{x_{2n_i} + 2\}$ and $\{x_{2n_i}\}$ of $\{x_{2n}\}$ and a number $L_0 \in [I, S]$ such that

$$\lim_{i\to\infty} x_{2n_i+2} = I \quad \text{and} \quad \lim_{i\to\infty} x_{2n_i} = L_0.$$

Note that when $A = 0$, then $\beta > 0$ and so $x_{n+1} > \beta > 0$ for $n \geq 0$. Therefore in all cases, $A + L_0 > 0$. From Eq. (1.1) we have

$$\frac{x_{2n+1}}{x_{2n-1}} = \frac{\alpha}{A + x_{2n}} \cdot \frac{1}{x_{2n-1}} + \frac{\beta x_{2n}}{A + x_{2n}} \cdot \frac{1}{x_{2n-1}} + \frac{\gamma}{A + x_{2n}} + \frac{\delta x_{2n-2}}{A + x_{2n}} \cdot \frac{1}{x_{2n-1}}$$

and

$$x_{2n+2} = \frac{\alpha}{A + x_{2n+1}} + \frac{\beta x_{2n+1}}{A + x_{2n+1}} + \frac{\gamma x_{2n}}{A + x_{2n+1}} + \frac{\delta x_{2n-1}}{A + x_{2n+1}}.$$

By replacing n by n_i in the above two identities and then by taking limits, as $i \to \infty$, we find

$$\lim_{i \to \infty} \left(\frac{x_{2n_i+1}}{x_{2n_i-1}} \right) = \frac{\gamma}{A + L_0}$$

and

$$I = \beta + \frac{\delta}{\gamma}(A + L_0).$$

Therefore,

$$I = \beta + \frac{\delta}{\gamma}(A + L_0) \leq L_0 \tag{2.3}$$

and so

$$L_0 \geq \frac{\beta\gamma + \delta A}{\gamma - \delta} \geq S.$$

Hence

$$L_0 = \frac{\beta\gamma + \delta A}{\gamma - \delta} = S \tag{2.4}$$

and so from Eqs. (2.3) and (2.4),

$$I = \beta + \frac{\delta}{\gamma}(A + L_0) = \frac{\beta\gamma + \delta A}{\gamma - \delta}.$$

The proof is complete. □

PERIOD-TWO CONVERGENCE

Our aim in this section is to show that when

$$\gamma = \beta + \delta + A$$

and

$$\beta + A > 0 \tag{3.1}$$

then every solution of Eq. (1.1), that is every solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + (\beta + \delta + A)x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots \tag{3.2}$$

converges to a (not necessarily prime) period-two solution of Eq. (3.2). The restriction (3.1) cannot be relaxed in this period-two convergence result. In fact, when $\beta = A = 0$ the resulting equation

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1} + \delta x_{n-2}}{x_n}, \quad n = 0, 1, \ldots \tag{3.3}$$

with

$$\alpha \geq 0 \quad \text{and} \quad \delta > 0$$

has unbounded solutions. See Refs. [2,7]. In particular, every solution of Eq. (3.3) with

$$x_{-2} = x_0 \leq \delta$$

is such that

$$x_{2n} = x_0, \quad \text{for } n \geq 0$$

and

$$x_{2n+1} = \frac{\delta}{x_0} x_{2n-1} + \left(\delta + \frac{\alpha}{x_0}\right) \to \infty, \quad \text{as } n \to \infty.$$

The proof of statement (b) of Theorem 1 is long and tedious and in order to simplify it, we first establish several lemmas describing the character of solutions of Eq. (3.2). We begin by stating several identities which follow from Eq. (3.2) and which will be used throughout this section. They are all valid for $n \geq 0$.

$$x_{2n+1} = \frac{\alpha + \beta x_{2n} + (\beta + \delta + A)x_{2n-1} + \delta x_{2n-2}}{A + x_{2n}} \tag{3.4a}$$

$$= \frac{\alpha}{A + x_{2n}} + \beta \frac{x_{2n}}{A + x_{2n}} + (\beta + \delta + A)\frac{x_{2n-1}}{A + x_{2n}} + \delta \frac{x_{2n-2}}{A + x_{2n}} \tag{3.4b}$$

$$x_{2n+2} = \frac{\alpha + \beta x_{2n+1} + (\beta + \delta + A)x_{2n} + \delta x_{2n-1}}{A + x_{2n+1}} \tag{3.5a}$$

$$= \frac{\alpha}{A + x_{2n+1}} + \beta \frac{x_{2n+1}}{A + x_{2n+1}} + (\beta + \delta + A)\frac{x_{2n}}{A + x_{2n+1}} + \delta \frac{x_{2n-1}}{A + x_{2n+1}} \tag{3.5b}$$

$$x_{n+2} - x_n = \frac{\beta + A}{A + x_{n+1}}(x_{n+1} - x_{n-1}) + \frac{\delta}{A + x_{n+1}}(x_n - x_{n-2}) \tag{3.6}$$

$$x_{2n+2} - x_{2n} = \frac{\beta + A}{A + x_{2n+1}}(x_{2n+1} - x_{2n-1}) + \frac{\delta}{A + x_{2n+1}}(x_{2n} - x_{2n-2}) \tag{3.7}$$

$$x_{2n+3} - x_{2n+1} = \frac{\beta + A}{A + x_{2n+2}}(x_{2n+2} - x_{2n}) + \frac{\delta}{A + x_{2n+2}}(x_{2n+1} - x_{2n-1}) \tag{3.8}$$

$$x_{2n+1} - x_{2n-1} = \frac{\alpha + \beta x_{2n}(\beta + \delta - x_{2n})x_{2n-1} + \delta x_{2n-2}}{A + x_{2n}} \tag{3.9}$$

$$x_{2n+2} - x_{2n} = \frac{\alpha + \beta x_{2n+1}(\beta + \delta - x_{2n+1})x_{2n} + \delta x_{2n-1}}{A + x_{2n+1}} \tag{3.10}$$

$$\frac{x_{2n+1}}{x_{2n-1}} = \frac{\alpha}{A + x_{2n}} \cdot \frac{1}{x_{2n-1}} + \beta \frac{x_{2n}}{A + x_{2n}} \cdot \frac{1}{x_{2n-1}} + \frac{\beta + \delta + A}{A + x_{2n}} + \delta \frac{x_{2n-2}}{A + x_{2n}} \cdot \frac{1}{x_{2n-1}} \tag{3.11}$$

$$\frac{x_{2n+2}}{x_{2n}} = \frac{\alpha}{A + x_{2n+1}} \cdot \frac{1}{x_{2n}} + \beta \frac{x_{2n+1}}{A + x_{2n+1}} \cdot \frac{1}{x_{2n}} + \frac{\beta + \delta + A}{A + x_{2n+1}} + \delta \frac{x_{2n-1}}{A + x_{2n+1}} \cdot \frac{1}{x_{2n}} \tag{3.12}$$

Among the above equations, the identity described by Eq. (3.6) is at the heart of the period-two convergence of solutions of Eq. (3.2). Its proof is a consequence of Eq. (3.2) as follows. Note that

$$x_{n+1}x_n = \alpha + \beta x_n + (\beta + \delta + A)x_{n-1} + \delta x_{n-2} - Ax_{n-1}$$

and so,

$$\begin{aligned} x_{n+2} - x_n &= \frac{\alpha + \beta x_{n+1} + (\beta + \delta + A)x_n + \delta x_{n-1}}{A + x_{n+1}} - x_n \\ &= \frac{\alpha + \beta x_{n+1} + (\beta + \delta)x_n + \delta x_{n-1} - \alpha - \beta x_n - (\beta + \delta + A)x_{n-1} - \delta x_{n-2} + Ax_{n+1}}{A + x_{n+1}} \\ &= \frac{(\beta + A)x_{n+1} - (\beta + A)x_{n-1} + \delta(x_n - x_{n-2})}{A + x_{n+1}} \\ &= \frac{\beta + A}{A + x_{n+1}}(x_{n+1} - x_{n-1}) + \frac{\delta}{A + x_{n+1}}(x_n - x_{n-2}). \end{aligned}$$

From Eq. (3.6), and more precisely from its equivalent versions (3.7) and (3.8), it is now clear that the following result is true about the subsequences of the even terms $\{x_{2n}\}_{n=-1}^{\infty}$ and the odd terms $\{x_{2n+1}\}_{n=-1}^{\infty}$ of every solution $\{x_n\}_{n=-2}^{\infty}$ of Eq. (3.2).

LEMMA (3.1) *The two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of every solution of Eq.* (3.2) *are either both eventually monotonically increasing, or they are both eventually monotonically decreasing, or one of them is monotonically increasing and the other is monotonically decreasing.*

In the sequel we will denote the limits of the subsequences of the even and odd terms of a solution of Eq. (3.2) by L_0 and L_1, respectively. That is,

$$L_0 = \lim_{n\to\infty} x_{2n} \quad \text{and} \quad L_1 = \lim_{n\to\infty} x_{2n+1}.$$

Each of these limits may have only one of the following three values:

$$0, \ \infty, \ \text{or a positive real number.}$$

Now let us look for all period-two solutions

$$\ldots, \phi, \ \psi, \ \ldots$$

of Eq. (3.2). From Eq. (3.2) we have

$$\phi = \frac{\alpha + \beta\psi + (\beta + \delta + A)\phi + \delta\psi}{A + \psi}$$

and so

$$\phi\psi = \alpha + (\beta + \delta)(\phi + \psi)$$

which implies that

$$\phi[\psi - (\beta + \delta)] = \alpha + (\beta + \delta)\psi$$

and

$$\psi[\phi - (\beta + \delta)] = \alpha + (\beta + \delta)\phi.$$

When $\phi \neq \psi$ we have a prime period-two solution, while when $\phi = \psi$ we see that ϕ is the equilibrium $\bar{x}$ of Eq. (3.2). Note also that in all cases

$$\bar{x}, \ \ \phi, \ \ \psi, \ \ \in (\beta + \delta, \infty)$$

provided that $\alpha + \beta + \delta > 0$.

For the sake of completeness and unification, our proof here of the period-two convergence of Eq. (3.2) includes all previous special cases of Eq. (3.2). When

$$\alpha = \beta = \delta = 0 \tag{3.13}$$

Eq. (3.2) reduces to

$$x_{n+1} = \frac{Ax_{n-1}}{A + x_n}, \quad \text{for } n = 0, 1, \ldots \tag{3.14}$$

with $A > 0$, from which it is clear that

$$x_{n+1} \leq x_{n-1}.$$

Therefore in this case, every solution of Eq. (3.14) converges to a (not necessarily prime) period-two solution of the form

$$\ldots, \phi, \ 0, \ \ldots \tag{3.15}$$

with $\phi \geq 0$. This completes the proof of Theorem 1(b) when (3.13) holds.

When $\delta = 0$, that is for the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + (\beta + A)x_{n-1}}{A + x_n}, \quad n = 0, 1, \ldots \tag{3.16}$$

it follows from Eq. (3.6) that

$$x_{n+2} - x_n = \frac{\beta + A}{A + x_{n+1}}(x_{n+1} - x_{n-1}), \quad \text{for } n \geq 0. \tag{3.17}$$

From Eq. (3.17) we see that for every solution of Eq. (3.16) exactly one of the following statements is true for all $n \geq 0$:

$$x_{n+1} < x_{n-1}$$
$$x_{n+1} = x_{n-1}$$
$$x_{n+1} > x_{n-1}.$$

Clearly all bounded solutions of Eq. (3.16) converge to a period-two solution. As in Ref. [11, p. 40], assume for the sake of contradiction that there is an unbounded solution, that is a solution such that:

$$\lim_{n \to \infty} x_{2n} = \infty$$

while $\{x_{2n+1}\}$ is increasing. The case where

$$\lim_{n \to \infty} x_{2n+1} = \infty$$

and $\{x_{2n}\}$ is increasing is similar and will be omitted. Choose $N \geq 0$ such that

$$\frac{\beta + A}{A + x_{2N+1}} \cdot \frac{\beta + A}{A + x_{2N}} < 1.$$

Define

$$\rho = \frac{\beta + A}{A + x_{2N+1}} \cdot \frac{\beta + A}{A + x_{2N}} \quad \text{and} \quad \sigma = (x_{2N} - x_{2N-2}).$$

Then

$$\frac{\beta + A}{A + x_{2n+1}} \cdot \frac{\beta + A}{A + x_{2n+1}} < \rho, \quad \text{for } n \geq N.$$

Hence from Eq. (3.17) we find

$$x_{2N+2} - x_{2N} = \frac{\beta + A}{A + x_{2N+1}} (x_{2N+1} - x_{2N-1})$$

$$= \frac{\beta + A}{A + x_{2N+1}} \cdot \frac{\beta + A}{A + x_{2N}} (x_{2N} - x_{2N-2}) < \sigma\rho.$$

It follows by induction that for $\mu = 1, 2, \ldots$

$$x_{2N+2\mu} - x_{2N+2(\mu-1)} < \sigma\rho^{\mu}$$

and so by summing up

$$x_{2N+2\mu} < x_{2N} + \frac{\sigma\rho}{1 - \rho}, \quad \text{for } \mu = 1, 2, \ldots.$$

This contradicts the assumption that the subsequence of the even terms converges to ∞, and completes the proof of Theorem 1(b) when $\delta = 0$. Therefore in the sequel we will assume that

$$\delta > 0.$$

Returning to the period-two solutions of Eq. (3.2) the following result is now clear.

LEMMA (3.2) *All prime period-two solutions*

$$\ldots, \phi, \ \psi, \ \ldots$$

of Eq. (3.2) *are given by*

$$\phi = \frac{(\beta + \delta)\psi + \alpha}{\psi - (\beta + \delta)}$$

with

$$\phi \neq \psi \quad \text{and} \quad \phi, \psi \in (\beta + \delta, \infty).$$

Clearly, when both L_0 and L_1 are positive numbers, the sequence

$$\ldots, L_0, \ L_1, \ \ldots$$

is a period-two solution of Eq. (3.2) and as we showed before

$$L_0 L_1 = \alpha + (\beta + \delta)(L_1 + L_2).$$

In particular,

$$L_0, L_1 \in (\beta + \delta, \infty).$$

When $L_0 = L_1$, then the solution converges to the equilibrium $\bar{x}$ of Eq. (3.2) and when $L_0 \neq L_1$ the solution converges to a prime period-two solution of Eq. (3.2). Note that Eq. (3.2) has a huge set of prime period-two solutions, as described by Lemma 3.2.

The following Lemma shows that neither L_0 nor L_1 can be zero.

LEMMA (3.3) *Neither of the subsequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *may converge to zero.*

Proof Assume for the sake of contradiction that

$$L_0 = 0.$$

The case where $L_1 = 0$ is similar and will be omitted. Now note that when $L_0 = 0$, A must be positive. Otherwise $A = 0$, $\beta > 0$, and Eq. (3.4b) implies that $x_{n+2} \geq \beta > 0$.

There are three possibilities for L_1: 0, ∞, or a positive number. We will show that each of them leads to a contradiction.

If $L_1 = 0$, then both subsequences are eventually decreasing to zero and Eq. (3.9), with n sufficiently large, implies that

$$0 \geq \frac{\alpha + \beta x_{2n} + (\beta + \delta - x_{2n})x_{2n-1} + \delta x_{2n-2}}{A + x_{2n+1}} > 0$$

which is impossible.

If $L_1 \in (0, \infty)$, by taking limits in Eq. (3.4a) we see that

$$L_1 = \frac{\alpha + (\beta + \delta + A)L_1}{A} > L_1$$

which is also impossible.

Finally if $L_1 = \infty$, by taking limits in Eq. (3.11) we find

$$\lim_{n\to\infty} \left(\frac{x_{2n+1}}{x_{2n-1}}\right) = \frac{\beta + \delta + A}{A}$$

and from Eq. (3.12) we obtain

$$1 \geq \left(\frac{x_{2n+2}}{x_{2n}}\right) \geq \delta \frac{1}{\frac{A}{x_{2n-1}} + \left(\frac{x_{2n+1}}{x_{2n-1}}\right)} \cdot \frac{1}{x_{2n}}$$

which leads to a contradiction, as $n \to \infty$. The proof is complete. □

It follows from Lemma 3.3 that L_0 and L_1 are positive numbers or ∞. The next results establish that neither L_0 nor L_1 may be below

$$\beta + \delta.$$

LEMMA (3.4)

$$L_0,\ L_1 \in [\beta + \delta, \infty].$$

Proof Assume for the sake of contradiction that

$$L_0 < \beta + \delta.$$

The proof when $L_1 \leq \beta + \delta$ is similar and will be omitted. There are two possibilities for L_1: it may be positive or ∞. We will show that each of them leads to a contradiction.

If $L_1 \in (0, \infty)$ then

$$\ldots, L_0,\ L_1,\ \ldots$$

is a period-two solution of Eq. (3.2) and so, by Lemma 4.2, $L_0 > \beta + \delta$, which is a contradiction. On the other hand, if $L_1 = \infty$, then from Eq. (3.11)

$$\lim_{n\to\infty}\left(\frac{x_{2n+1}}{x_{2n-1}}\right) = \frac{\beta+\delta+A}{A+L_0}$$

and so from Eq. (3.12),

$$1 = \lim_{n\to\infty}\left(\frac{x_{2n+2}}{x_{2n}}\right) = \frac{\beta}{L_0} + \delta\frac{1}{0+\frac{\beta+\delta+A}{A+L_0}}\cdot\frac{1}{L_0}$$

that is

$$L_0 = \beta + \delta$$

which is also a contradiction. □

LEMMA (3.5)

(i) *If* $L_0 \in (0, \infty)$ *and* $L_1 = \infty$, *then* $L_0 = \beta + \delta$.
(ii) *If* $L_0 = \infty$ *and* $L_1 \in (0, \infty)$, *then* $L_1 = \beta + \delta$.

Proof We will prove (i). The proof of (ii) is similar and will be omitted. From Eq. (3.11) we obtain

$$\lim_{n\to\infty}\left(\frac{x_{2n+1}}{x_{2n-1}}\right) = \frac{\beta+\delta+A}{A+L_0}$$

and so from Eq. (3.12),

$$1 = \frac{\beta}{L_0} + \frac{\delta}{L_0}\cdot\frac{A+L_0}{\beta+\delta+A}.$$

Hence

$$L_0 = \beta + \delta$$

and the proof is complete. □

LEMMA (3.6) *It is not possible that both* L_0 *and* L_1 *are equal to* ∞.

Proof Otherwise from Eq. (3.11)

$$1 \le \lim_{n\to\infty}\left(\frac{x_{2n+1}}{x_{2n-1}}\right) = 0$$

which is impossible. □

LEMMA (3.7) *Every solution of Eq.* (3.2) *is eventually bounded from below by* $(\beta + \delta)$.

Proof Assume for the sake of contradiction that there exists a solution of Eq. (3.2) which is not bounded from below by $(\beta + \delta)$. Then in view of the previous lemmas the only thing that the solution can do is that one of the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ eventually increases to

$(\beta + \delta)$ while the other eventually increases to ∞. We will assume

$$\lim_{n\to\infty} x_{2n} = \beta + \delta, \quad \lim_{n\to\infty} x_{2n+1} = \infty$$

with both subsequences being eventually increasing. The case where the even and odd subsequences are interchanged is similar and will be omitted.

Let $\epsilon \in (0, \beta + \delta)$ be given and sufficiently small, and let $N \geq 0$ be such that

$$\beta + \delta - \epsilon < x_{2n} < \beta + \delta \quad \text{for } n \geq N.$$

Then for any $N_0 \geq N$ and sufficiently large we have

$$x_{2N_0+2} < \beta + \delta$$

which implies that

$$x_{2N_0+2} = \frac{\alpha + \beta x_{2N_0+1} + (\beta + \delta + A)x_{2N_0} + \delta x_{2N_0-1}}{A + x_{2N_0+1}} < \beta + \delta. \tag{3.18}$$

Define

$$R_0 = \frac{(\beta + \delta + A)(\beta + \delta - \epsilon) + \alpha - (\beta + \delta)A}{\delta}.$$

Then Eq. (3.18) implies that

$$\alpha + \beta x_{2N_0+1} + (\beta + \delta + A)x_{2N_0} + \delta x_{2N_0-1} < (\beta + \delta)A + (\beta + \delta)x_{2N_0+1}$$

and so

$$\begin{aligned} x_{2N_0+1} &> x_{2N_0-1} + \frac{\beta + \delta + A}{\delta} x_{2N_0} + \frac{\alpha - (\beta + \delta)A}{\delta} \\ &> x_{2N_0-1} + \frac{(\beta + \delta + A)}{\delta}(\beta + \delta - \epsilon) + \frac{\alpha - (\beta + \delta)A}{\delta}. \end{aligned}$$

Hence

$$x_{2N_0+1} > x_{2N_0-1} + R_0$$

and by using Eq. (3.2),

$$\frac{\alpha + \beta x_{2N_0} + (\beta + \delta + A)x_{2N_0-1} + \delta x_{2N_0-2}}{A + x_{2N_0}} > x_{2N_0-1} + R_0.$$

Therefore,

$$\alpha + \beta x_{2N_0} + (\beta + \delta + A)x_{2N_0-1} + \delta x_{2N_0-2} > Ax_{2N_0-1} + x_{2N_0}x_{2N_0-1} + R_0(A + x_{2N_0})$$

and so

$$(\beta + \delta - x_{2N_0})x_{2N_0-1} > R_0(A + x_{2N_0}) - \alpha - \beta x_{2N_0} - \delta x_{2N_0-2}$$

that is,

$$x_{2N_0-1} > \frac{R_0(A + x_{2N_0}) - \alpha - \beta x_{2N_0} - \delta x_{2N_0-2}}{\beta + \delta - x_{2N_0}}.$$

Hence

$$(\beta+\delta)x_{2N_0-1} - x_{2N_0}x_{2N_0-1} > R_0(A + x_{2N_0}) - \alpha - \beta x_{2N_0} - \delta x_{2N_0}$$

or equivalently,

$$(\beta+\delta)x_{2N_0-1} - \alpha - \beta x_{2N_0-1} - (\beta+\delta+A)x_{2N_0-2} - \delta x_{2N_0-3} + Ax_{2N_0}$$
$$> R_0(A + x_{2N_0}) - \alpha - \beta x_{2N_0} - \delta x_{2N_0-2}.$$

Thus,

$$x_{2N_0-1} > x_{2N_0-3} + \frac{R_0}{\delta}(A + x_{2N_0}) + \frac{1}{\delta}(\beta+A)(x_{2N_0-2} - x_{2N_0})$$
$$> x_{2N_0-3} + \frac{R_0}{\delta}(A + \beta + \delta - \epsilon) - \frac{1}{\delta}(\beta+A)(\beta+\delta)$$

and so from Eq. (3.2) we see that

$$\frac{\alpha + \beta x_{2N_0-2} + (\beta+\delta+A)x_{2N_0-3} + \delta x_{2N_0-4}}{A + x_{2N_0-2}}$$
$$> x_{2N_0-3} + \frac{R_0}{\delta}(A + \beta + \delta - \epsilon) - \frac{(\beta+A)(\beta+\delta)}{\delta}.$$

Therefore,

$$x_{2N_0-3} > \frac{R_1(A + x_{2N_0-2}) - \alpha - \beta x_{2N_0-2} - \delta x_{2N_0-4}}{\beta + \delta - x_{2N_0-2}}$$

where

$$R_1 = \frac{\beta+\delta+A-\epsilon}{\delta}R_0 - \frac{(\beta+A)(\beta+\delta)}{\delta}.$$

It follows by induction that for $k \geq 0$,

$$x_{2N_0-(2k+1)} > \frac{R_k(A + x_{2N_0-2k}) - \alpha - \beta x_{2N_0-2k} - \delta x_{2N_0-2(k+1)}}{\beta + \delta - x_{2N_0-2k}}$$

with

$$R_{k+1} = \frac{\beta+\delta+A-\epsilon}{\delta}R_k - \frac{(\beta+A)(\beta+\delta)}{\delta}.$$

Clearly for N_0 and k sufficiently large, this leads to a contradiction and the proof of the lemma is complete. □

We are now ready to present the proof of Theorem 1(b).

Proof (*of Theorem* 1(*b*)) Clearly every bounded solution of Eq. (3.2) converges to a (not necessarily prime) period-two solution. So assume for the sake of contradiction that Eq. (3.2) has an unbounded solution. We will assume that

$$\lim_{n\to\infty} x_{2n+1} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_{2n} = \beta + \delta,$$

with the subsequence of even terms of the solution being eventually decreasing and the subsequence of odd terms being eventually increasing. The case where the behavior of the even and odd subsequences is reversed is similar and will be omitted.

Then from Eq. (3.6) we obtain

$$x_{2n+3} - x_{2n+1} < \frac{\delta}{A + x_{2n+2}}(x_{2n+1} - x_{2n-1})$$

$$< \frac{\delta}{\beta + \delta + A}(x_{2n+1} - x_{2n-1}), \quad \text{for } n \geq 0.$$

Therefore,

$$x_{2n+1} - x_{2n-1} < \frac{\delta}{(\beta + \delta + A)^n}(x_1 - x_{-1})$$

and by summing up we find

$$x_{2n+1} - x_1 < \frac{\delta}{\beta + A}.$$

This contradicts the hypothesis that

$$\lim_{n\to\infty} x_{2n+1} = \infty$$

and completes the proof of the theorem. □

References

[1] A. M. Amleh, V. Kirk and G. Ladas, On the dynamics of $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{A+Bx_{n-2}}$, *Math. Sci. Res. Hot-Line* **5** (2001), 1–15.

[2] E. Camouzis, On rational third order difference equations, *Proceedings of the Eighth International Conference on Difference Equations and Applications*, July 28–Aug 2, 2003, Brno, Czech Republic (to appear).

[3] E. Camouzis, R. DeVault and W. Kosmala, On the period-two trichotomy of all positive solutions of $x_{n+1} = \frac{p+x_{n-2}}{x_n}$, *J. Math. Anal. Appl.* **291** (2004), 40–49.

[4] E. Camouzis and G. Ladas, Three trichotomy conjectures, *J. Differ. Equ. Appl.* **8** (2002), 495–500.

[5] E. Camouzis, G. Ladas and H. D. Voulov, On the dynamics of $x_{n+1} = \frac{\alpha+\gamma x_{n-1}+\delta x_{n-2}}{A+x_{n-2}}$, *J. Differ. Equ. Appl.* **9** (2003), 731–738.

[6] E. Chatterjee, E. A. Grove, Y. Kostrov and G. Ladas, On the trichotomy character of $x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Bx_n+Dx_{n-2}}$, *J. Differ. Equ. Appl.* **9** (2003), 1113–1128.

[7] C.A. Clark, M.R.S. Kulenovic, and S. Valicenti, *On the dynamics of* $x_{n+1} = (\alpha x_{n-1} + \beta x_{n-2})/(A + x_n)$, *Math. Sci. Res. Journal* (to appear).

[8] C. H. Gibbons, M. R. S. Kulenovic and G. Ladas, On the recursive sequence $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$, *Math. Sci. Res. Hot-Line* **4**(2) (2002), 1–11.

[9] C. H. Gibbons, M. R. S. Kulenovic, G. Ladas and H. D. Voulov, On the trichotomy character of $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+x_n}$, *J. Differ. Equ. Appl.* **8** (2002), 75–92.

[10] E. A. Grove, G. Ladas, M. Predescu and M. Radin, On the global character of $x_{n+1} = \frac{px_{n-1}+x_{n-2}}{q+x_{n-2}}$, *Math. Sci. Res. Hot-Line* **5**(7) (2001), 25–39.

[11] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations With Open Problems and Conjectures, Chapman & Hall/CRC Press, 2001.