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Open Problems and Conjectures

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In this section, we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas. Email: gladas@math.uri.edu

Keywords: Boundedness; Convergence; Stability; Periodicity

On second-order rational difference equations, Part 2

A. M. AMLEH†, E. CAMOUZIS‡ and G. LADAS¶*

†Department of Mathematics and Computing Science, Saint Mary's University, 923 Robie Street, Halifax, Nova Scotia, Canada B3H 3C3
‡Department of Mathematics and Natural Sciences, American College of Greece, 6 Gravias Street, Aghia Paraskevi, 15342 Athens, Greece
¶Department of Mathematics, University of Rhode Island, Kingston, RI 02881-0816, USA

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1. Introduction

This is Part 2 of our paper [3] which deals with the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{1.1}$$

with nonnegative parameters α, β, γ, A, B, C and with arbitrary nonnegative initial conditions x_{-1}, x_0 such that the denominator is always positive. Some extensions and generalizations of equation (1.1) are also considered here. For some related results see [1–52].

As we mentioned in Part 1 of Ref. [3], equation (1.1) contains 28 nontrivial special cases whose character is summarized in the following table. Please see the Appendix A for the meaning of the abbreviations: **ESC**$\bar{x}$, **ESC**, **ESP**$_k$, **ESCP**$_k$, etc.

Summary of the behaviour of the 28 nontrivial second-order rational difference equations.

Confirm or refute each one of the eight conjectures on the previous table.

*Corresponding author. Email: gladas@math.uri.edu

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Summary of the behaviour of the 28 nontrivial second-order rational difference equations.

#20: $x_{n+1}=\frac{\alpha}{Bx_n+x_{n-1}}$	**ESC**$\bar{x}$				
#24: $x_{n+1}=\frac{\beta x_n}{1+x_{n-1}}$	**ESC**				
Pielou's equation					
#26: $x_{n+1}=\frac{x_n}{Bx_n+x_{n-1}}$	**ESC**$\bar{x}$				
#29: $x_{n+1}=\frac{x_{n-1}}{A+x_n}$			$\mathbf{P_2-Tricho}$		
#32: $x_{n+1}=\frac{x_{n-1}}{Bx_n+x_{n-1}}$		$\mathbf{ESCP_2}$			
#43: $x_{n+1}=\frac{\alpha+x_n}{x_{n-1}}$					
Lyness's equation					
#46: $x_{n+1}=\frac{\alpha+x_{n-1}}{x_n}$			$\mathbf{P_2-Tricho}$		
#54: $x_{n+1}=\beta+\frac{x_{n-1}}{x_n}$			**First** $\mathbf{P_2-Tricho}$		
#55: $x_{n+1}=\gamma+\frac{x_n}{x_{n-1}}$	**ESC**$\bar{x}$				
#66: $x_{n+1}=\frac{\alpha+x_n}{A+x_{n-1}}$				**Conjecture: ESC**$\bar{x}$	
#68: $x_{n+1}=\frac{\alpha+x_n}{x_n+Cx_{n-1}}$				**Conjecture: ESC**$\bar{x}$	
#71: $x_{n+1}=\frac{\alpha+x_{n-1}}{A+x_n}$			$\mathbf{P_2-Tricho}$		
#74: $x_{n+1}=\frac{\alpha+x_{n-1}}{Bx_n+x_{n-1}}$		$\mathbf{ESCP_2}$			
#83: $x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}}{A+x_n}$			$\mathbf{P_2-Tricho}$		
#84: $x_{n+1}=\frac{\beta x_n+x_{n-1}}{A+x_{n-1}}$	**ESC**				
#86: $x_{n+1}=\frac{\beta x_n+\gamma x_{n-1}}{Bx_n+x_{n-1}}$		$\mathbf{ESCP_2}$			
#101: $x_{n+1}=\frac{1}{1+Bx_n+Cx_{n-1}}$	**ESC**$\bar{x}$				
#105: $x_{n+1}=\frac{\beta x_n}{A+x_n+Cx_{n-1}}$	**ESC**				
#109: $x_{n+1}=\frac{x_{n-1}}{A+Bx_n+x_{n-1}}$		$\mathbf{ESCP_2}$			
#118: $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{x_n}$			$\mathbf{P_2-Tricho}$		
#119: $x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}}{x_{n-1}}$				**Conjecture: ESC**$\bar{x}$	
#141: $x_{n+1}=\frac{\alpha+x_n}{A+Bx_n+x_{n-1}}$				**Conjecture: ESC**$\bar{x}$	
#145: $x_{n+1}=\frac{\alpha+x_{n-1}}{A+Bx_n+x_{n-1}}$		$\mathbf{ESCP_2}$			
#153: $x_{n+1}=\frac{\beta x_n+x_{n-1}}{A+Bx_n+x_{n-1}}$					**Conjecture:** $\mathbf{ESCP_2}$
#165: $x_{n+1}=\frac{\alpha+\beta x_n+x_{n-1}}{A+x_n}$			$\mathbf{P_2-Tricho}$		
#166: $x_{n+1}=\frac{\alpha+x_n+\gamma x_{n-1}}{A+x_{n-1}}$				**Conjecture: ESC**$\bar{x}$	
#168: $x_{n+1}=\frac{\alpha+x_n+\gamma x_{n-1}}{Bx_n+x_{n-1}}$					**Conjecture:** $\mathbf{ESCP_2}$
#201: $x_{n+1}=\frac{\alpha+\beta x_n+x_{n-1}}{A+Bx_n+x_{n-1}}$					**Conjecture:** $\mathbf{ESCP_2}$

Confirm or refute each one of the eight conjectures on the previous table.

2. For Equation (#86), $ESCP_2$

This equation was investigated in Refs. [36,37,39] and [48]. Equation (#86) can be written in the normalized form,

$$x_{n+1} = \frac{\beta x_n + x_{n-1}}{Bx_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{2.1}$$

with positive parameters β, B and with arbitrary positive initial conditions x_{-1}, x_0.

The only equilibrium of equation (2.1) is

$$\bar{x} = \frac{\beta + 1}{B + 1}.$$

The characteristic equation of the linearized equation of equation (2.1) about the equilibrum is

$$\lambda^2 - \frac{\beta - B}{(\beta + 1)(B + 1)}\lambda + \frac{\beta - B}{(\beta + 1)(B + 1)} = 0.$$

From this it follows that the positive equilibrium $\bar{x}$ of equation (2.1) is locally asymptotically stable when

$$\beta \geq B \tag{2.2}$$

or

$$\beta < B \quad \text{and} \quad B < 3\beta + \beta B + 1 \tag{2.3}$$

and unstable (saddle point) when

$$B > 3\beta + \beta B + 1. \tag{2.4}$$

When (2.4) holds, and only then, equation (2.1) possesses the unique prime period-two solution

$$\ldots, \frac{1 - \beta - \sqrt{(1-\beta)^2 - (4\beta(1-\beta))/B - 1}}{2}, \frac{1 - \beta + \sqrt{(1-\beta)^2 - (4\beta(1-\beta))/B - 1}}{2}, \ldots \tag{2.5}$$

which is locally asymptotically stable. For the proof of this, see Ref. [36].

In the next theorem we present the global character of solutions of Equation (2.1).

THEOREM 2.1. *The following statements are true:*

(a) *The equilibrium $\bar{x}$ of equation* (2.1) *is globally asymptotically stable when* (2.2) *or* (2.3) *holds.*

(b) *Every solution of equation* (2.1) *converges to the equilibrium $\bar{x}$ of equation* (2.1) *when*

$$B = 3\beta + \beta B + 1. \tag{2.6}$$

(c) *Every solution of equation* (2.1) *converges to a* (*not necessarily prime*) *period-two solution when* (2.4) *holds.*

Proof. Let $\{x_n\}$ be a solution of equation (2.1) and assume that (2.2) holds. For all $n \geq 0$,

$$1 \leq x_{n+1} = \frac{\beta x_n + x_{n-1}}{Bx_n + x_{n-1}} = \frac{\beta}{B} \times \frac{\beta B x_n + B x_{n-1}}{\beta B x_n + \beta x_{n-1}} \leq \frac{\beta}{B}$$

which implies that the interval $[1, (\beta/B)]$ is invariant for the solution $\{x_n\}$. Furthermore, the solution $\{x_n\}$ satisfies the following equation:

$$\begin{aligned} x_{n+1} &= \frac{\beta(\beta x_{n-1} + x_{n-2})/(Bx_{n-1} + x_{n-2}) + x_{n-1}}{B(\beta x_{n-1} + x_{n-2})/(Bx_{n-1} + x_{n-2}) + x_{n-1}} = F(x_{n-1}, x_{n-2}) \\ &= \frac{\beta^2 x_{n-1} + Bx_{n-1}^2 + \beta x_{n-2} + x_{n-1}x_{n-2}}{\beta B x_{n-1} + Bx_{n-1}^2 + Bx_{n-2} + x_{n-1}x_{n-2}}. \end{aligned} \tag{2.7}$$

Clearly

$$F \in C\left(\left[1, \frac{\beta}{B}\right]^2, \left[1, \frac{\beta}{B}\right]\right),$$

and

$$F_{x_{n-1}} = \frac{(B-\beta)\beta B x_{n-1}^2 + 2\beta B(B-\beta)x_{n-1}x_{n-2} + (B-\beta)x_{n-2}^2}{(\beta B x_{n-1} + Bx_{n-1}^2 + Bx_{n-2} + x_{n-1}x_{n-2})^2} \leq 0,$$

and

$$F_{x_{n-2}} = \frac{-x_{n-1}^2(B-\beta)^2}{(\beta B x_{n-1} + Bx_{n-1}^2 + Bx_{n-2} + x_{n-1}x_{n-2})^2} \leq 0,$$

and for each $m, M \in [1, (\beta/B)]$, the system

$$M = \frac{\beta^2 m + Bm^2 + \beta m + m^2}{\beta B m + Bm^2 + Bm + m^2} \quad \text{and} \quad m = \frac{\beta^2 M + BM^2 + \beta M + M^2}{\beta BM + BM^2 + BM + M^2}$$

has the unique solution $(m, M) = (\bar{x}, \bar{x})$. By employing Theorem 2.4 from Part 1 the result follows.

On the other hand assume that

$$\beta < B.$$

Clearly the function

$$\frac{\beta x_n + x_{n-1}}{Bx_n + x_{n-1}}$$

is strictly decreasing in x_n and strictly increasing in x_{n-1}. By employing Theorem 2.5 in Part 1, we find that the solution $\{x_n\}$ converges to a (not necessarily prime) period-two solution. Due to the fact that equation (2.1) possesses a prime period-two solution only when (2.4) holds, (*a*), (*b*) and (*c*) follow. The proof is complete. □

Open Problem 2.1. *Assume that* (2.4) *holds.*

(i) *Determine the set of initial conditions* x_{-1}, x_0 *for which every solution of equation* (2.1) *converges to the equilibrium* $\bar{x}$.

(ii) *Determine the set of initial conditions* x_{-1}, x_0 *for which every solution of equation* (2.1) *converges to* (2.5).

3. An important identity for Equation (#141)

Equation (#141) can be written in the normalized form,

$$x_{n+1} = \frac{\alpha + x_n}{A + Bx_n + x_{n-1}}, n = 0, 1, \ldots \tag{3.1}$$

with positive parameters α, A, B and with arbitrary nonnegative initial conditions x_{-1}, x_0.

Here we present an important identity for equation (3.1). Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of equation (3.1). Then the following identity is true:

$$\begin{aligned} x_{n+1} - x_{n-3} = & \frac{(\alpha A - A^2 x_{n-3} - A x_{n-3}^2) + x_n(A + x_{n-3})(1 - Bx_{n-3})}{(A + Bx_n)(A + Bx_{n-2} + x_{n-3}) + \alpha + x_{n-2}} \\ & + \frac{x_{n-2}(\alpha B - (AB+1)x_{n-3}) + x_n x_{n-2} B(1 - Bx_{n-3})}{(A + Bx_n)(A + Bx_{n-2} + x_{n-3}) + \alpha + x_{n-2}} \end{aligned} \tag{3.2}$$

Note that

$$x_{N-3} \geq \frac{1}{B} \quad \text{and} \quad \alpha < \frac{A}{B} + \frac{1}{B^2} \rightarrow \begin{cases} \alpha - Ax_{N-3} - x_{N-3}^2 < 0 \\ 1 - Bx_{N-3} \leq 0 \\ \alpha B - (AB+1)x_{N-3} < 0 \end{cases}$$

and

$$x_{N-3} \leq \frac{1}{B} \quad \text{and} \quad \alpha > \frac{A}{B} + \frac{1}{B^2} \rightarrow \begin{cases} \alpha - Ax_{N-3} - x_{N-3}^2 > 0 \\ 1 - Bx_{N-3} \geq 0 \\ \alpha B - (AB+1)x_{N-3} > 0. \end{cases}$$

THEOREM 3.1. *Let* $\{x_n\}$ *be any solution of equation* (3.1). *Then the following statements are true:*

(i) *When*

$$0 < A < 1 \quad \textit{and} \quad \frac{(1-B)(1-A)^2}{4B^2} \leq \alpha < \frac{A}{B} + \frac{1}{B^2} \tag{3.3}$$

then the solution $\{x_n\}$ *eventually enters the interval* $[\alpha B - A, (1/B)]$ *and the function*

$$f(x_n, x_{n-1}) = \frac{\alpha + x_n}{A + Bx_n + x_{n-1}}$$

is eventually strictly increasing in x_n *and strictly decreasing in* x_{n-1}. *Furthermore, the solution converges to the equilibrium.*

(ii) *When*

$$0 < A < 1 \quad and \quad \alpha > \frac{A}{B} + \frac{1}{B^2} \tag{3.4}$$

the solution $\{x_n\}$ *eventually enters the interval* $[1/B,\ \alpha B - A]$ *and the function* $f(x_n, x_{n-1})$ *is eventually strictly decreasing in* x_n *and* x_{n-1}. *Furthermore, the solution converges to the equilibrium.*

(iii) *When*

$$0 < A < 1 \quad and \quad \alpha = \frac{A}{B} + \frac{1}{B^2} \tag{3.5}$$

then the solution $\{x_n\}$ *converges to the equilibrium.*

Proof. Let $\{x_n\}$ be a solution of equation (3.1) with nonnegative initial conditions. We claim that

$$\left[\min\left(\alpha B - A, \frac{1}{B}\right), \max\left(\alpha B - A, \frac{1}{B}\right)\right]$$

is an attracting interval for the solution $\{x_n\}$ of equation (3.1).

We will prove that when (3.3) or (3.4) holds all four subsequences of the solution $\{x_n\}$, of the form $\{x_{4n+j}\}_{j=0}^{3}$, lie eventually within the interval

$$\left[\min\left(\alpha B - A, \frac{1}{B}\right), \max\left(\alpha B - A, \frac{1}{B}\right)\right].$$

We will give the proof when (3.3) holds. The proof when (3.4) is similar and will be omitted. Furthermore, we will give the proof for the subsequence $\{x_{n+1}\}$. The proof for all the other subsequences is similar and will be omitted.

Suppose for the sake of contradiction that there exists N sufficiently large such that

$$x_{4N+1} < \alpha B - A \quad \text{or} \quad x_{4N+1} > \frac{1}{B}.$$

We will give the proof in the case where $x_{4N+1} < \alpha B - A$. The proof in the other case is similar and will be omitted. Then from

$$x_{4N+1} < \alpha B - A$$

it follows that

$$x_{4N+3} = \frac{\alpha + x_{4N+2}}{A + Bx_{4N+2} + x_{4N+1}} > \frac{\alpha + x_{4N+2}}{A + Bx_{4N+2} + \alpha B - A} = \frac{1}{B} > \alpha B - A$$

and so

$$x_{4N+5} = \frac{\alpha + x_{4N+4}}{A + Bx_{4N+4} + x_{4N+3}} < \frac{\alpha + x_{4N+4}}{A + Bx_{4N+4} + \alpha B - A} = \frac{1}{B}. \tag{3.6}$$

We claim that for some $k \geq 1$,

$$x_{4N+4k+1} \geq \alpha B - A. \tag{3.7}$$

Otherwise for all $k \geq 1$,

$$x_{4N+4k+1} < \alpha B - A.$$

Then clearly for all $k \geq 1$,

$$x_{4N+4k+3} = \frac{\alpha + x_{4N+4k+2}}{A + Bx_{4N+4k+2} + x_{4N+4k+1}} > \frac{\alpha + x_{4N+4k+2}}{A + Bx_{4N+4k+2} + \alpha B - A} = \frac{1}{B}.$$

From (3.2) it follows that the subsequence $\{x_{4N+4k+1}\}$ decreases. By taking limits in (3.2) we get a contradiction which proves (3.7). Assume without loss of generality that (3.7) holds for $k = 1$. From this and (3.6) we see that

$$\alpha B - A < x_{4N+5} < \frac{1}{B}.$$

Then

$$x_{4N+7} = \frac{\alpha + x_{4N+6}}{A + Bx_{4N+6} + x_{4N+5}} < \frac{\alpha + x_{4N+6}}{A + Bx_{4N+6} + \alpha B - A} < \frac{1}{B}$$

and

$$x_{4N+7} = \frac{\alpha + x_{4N+6}}{A + Bx_{4N+6} + x_{4N+5}} > \frac{\alpha + x_{4N+6}}{A + Bx_{4N+6} + (1/B)} > \frac{\alpha}{A + (1/B)} > \alpha B - A$$

and the result follows by induction.

When (3.3) holds, and due to the fact that the solution $\{x_n\}$ eventually enters the interval $[\alpha B - A,\ 1/B]$, we see that the function

$$f(x_n, x_{n-1}) = \frac{\alpha + x_n}{A + Bx_n + x_{n-1}}$$

is eventually strictly increasing in x_n and strictly decreasing in x_{n-1}. Furthermore for each m, $M \in [1,(\alpha B - A,\ 1/B)]$, in view of (3.3), the system

$$M = \frac{\alpha + M}{A + BM + m} \quad \text{and} \quad m = \frac{\alpha + m}{A + Bm + M}$$

has a unique solution $(m, M) = (\bar{x}, \bar{x})$. Hence, the result follows by Theorem 2.4 from Part 1.

When (3.4) holds, and due to the fact that the solution $\{x_n\}$ eventually enters the interval $[1/B,\ \alpha B - A]$, we see that the function

$$f(x_n, x_{n-1}) = \frac{\alpha + x_n}{A + Bx_n + x_{n-1}}$$

is strictly decreasing in x_n and eventually strictly decreasing in x_{n-1}. Furthermore, for each m,$M \in [1/B,\ \alpha B - A]$, the system

$$M = \frac{\alpha + m}{A + (B+1)m} \quad \text{and} \quad m = \frac{\alpha + M}{A + (B+1)M}$$

has a unique solution $(m, M) = (\bar{x}, \bar{x})$. Hence, the result follows by Theorem 2.4 from Part 1.

Finally, assume that (3.5) holds. Then clearly for all $n \geq 0$,

$$x_{n+1} - \frac{1}{B} = \frac{1}{B} \times \frac{(1/B) - x_{n-1}}{A + Bx_n + x_{n-1}}$$

from which it follows that each one of the four subsequences $\{x_{4n+j}\}, j \in \{0, 1, 2, 3\}$ is either above $1/B$, or below $1/B$, or identically equal to $1/B$. In view of (3.2) all four subsequences converge monotonically to finite limits. In addition from (3.2) we see that for all $n \geq 3$,

$$x_{n+1} = x_{n-3} \quad \text{if and only if} \quad x_{n-3} = \frac{1}{B}.$$

Hence all four subsequences converge to $1/B$. The proof is complete. □

The following theorem extends the result of Theorem 3.1 to the more general rational equation

$$x_{n+1} = \frac{\alpha + x_{n-m}}{A + Mx_{n-m} + Lx_{n-l}}, \quad n = 0, 1, \ldots \tag{3.8}$$

with $l, m \in \{0,1,\ldots\}$, with positive parameters α, A, M, L and with arbitrary nonnegative initial conditions.

The proof, as in the case of Theorem 3.1, is based on the following identity:

$$x_{n+1} - x_{n-2l-1} = \frac{\left(\alpha A - A^2 x_{n-2l-1} - ALx_{n-2l-1}^2\right) + x_{n-m}(A + Lx_{n-2l-1})(1 - Mx_{n-2l-1})}{(A + Mx_{n-m})(A + Mx_{n-l-m-1} + Lx_{n-2l-1}) + L\alpha + Lx_{n-l-m-1}} + \frac{x_{n-l-m-1}(\alpha M - (AM + L)x_{n-2l-1}) + x_{n-m}x_{n-l-m-1}M(1 - Mx_{n-2l-1})}{(A + Mx_{n-m})(A + Mx_{n-l-m-1} + Lx_{n-2l-1}) + L\alpha + Lx_{n-l-m-1}}. \tag{3.9}$$

THEOREM 3.2. *Let $\{x_n\}$ be any solution of equation* (3.8). *Then the following statements are true:*

(i) *When*

$$0 < A < 1 \quad \text{and} \quad \frac{(L - M)(1 - A)^2}{4M^2} \leq \alpha < \frac{A}{M} + \frac{L}{M^2} \tag{3.10}$$

the solution $\{x_n\}$ eventually enters the interval $[(\alpha M - A)/L,\ 1/M]$ and the function $f(x_{n-m}, x_{n-l})$ is eventually strictly increasing in $\{x_{n-m}\}$ and strictly decreasing in $\{x_{n-l}\}$. Furthermore, the solution converges to the equilibrium.

(ii) *When*

$$0 < A < 1 \quad \text{and} \quad \alpha > \frac{A}{M} + \frac{L}{M^2} \tag{3.11}$$

the solution $\{x_n\}$ eventually enters the interval $[1/M,\ (\alpha M - A)/L]$ and the function $f(x_{n-m}, x_{n-l})$ is eventually strictly decreasing in x_{n-m} and x_{n-l}. Furthermore, the solution converges to the equilibrium.

(iii) *When*

$$0 < A < 1 \quad \text{and} \quad \alpha = \frac{A}{M} + \frac{L}{M^2} \tag{3.12}$$

the solution $\{x_n\}$ converges to the equilibrium.

Proof. The proof is similar to the proof of Theorem 3.1 and will be omitted. □

4. Equation (#145)

This equation was investigated in Ref. [36]. Equation (#145) can be written in the normalized form,

$$x_{n+1} = \frac{\alpha + x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{4.1}$$

with positive parameters α, A, B and with arbitrary nonnegative initial conditions x_{-1}, x_0.

Equation (4.1) has the unique equilibrium

$$\bar{x} = \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha(1+B)}}{2(1+B)}.$$

The characteristic equation of the linearized equation about the equilibrium is

$$\lambda^2 + \frac{B\bar{x}}{A + (1+B)\bar{x}}\lambda + \frac{\bar{x} - 1}{A + (1+B)\bar{x}} = 0.$$

From this it follows that the positive equilibrium is locally asymptotically stable when

$$\bar{x} > \frac{1-A}{2}$$

which is equivalent to

$$A \geq 1, \tag{4.2}$$

or

$$A < 1 \quad \text{and} \quad B \leq 1, \tag{4.3}$$

or

$$A < 1,\, B > 1, \quad \text{and} \quad \alpha > \frac{(B-1)(1-A)^2}{4}, \tag{4.4}$$

and unstable (saddle point) when

$$A < 1,\, B > 1, \quad \text{and} \quad \alpha < \frac{(B-1)(1-A)^2}{4}. \tag{4.5}$$

By Theorems 2.8 and 2.10 from Part 1 it follows that when

$$A \geq 1$$

the equilibrium of equation (4.1) is globally asymptotically stable.

By Theorem 3.2 it follows that when

$$0 < A < 1 \quad \text{and} \quad \alpha \geq \frac{(B-1)(1-A)^2}{4} \tag{4.6}$$

every solution of equation (4.1) converges to the equilibrium.

When (4.5) holds, equation (4.1) has the unique prime period-two solution

$$\ldots, \frac{1-A-\sqrt{(1-A)^2-4\alpha/B-1}}{2}, \frac{1-A+\sqrt{(1-A)^2-4\alpha/(B-1)}}{2}, \ldots \tag{4.7}$$

which is locally asymptotically stable. See Ref. [36].

The following theorem establishes the global behaviour of solutions of equation (4.1) when (4.5) holds.

Theorem 4.1. *Assume that* (4.5) *holds. Then every solution of equation* (4.1) *converges to a* (*not necessarily prime*) *period-two solution.*

Proof. Let $\{x_n\}$ be a solution of equation (4.1). Due to the fact that

$$\frac{(B-1)(1-A)^2}{4} < B+A$$

it follows from (4.5) that

$$\alpha < B+A$$

and from Theorem 3.2 (*i*) it follows that the function

$$f(x_n, x_{n-1}) = \frac{\alpha + x_{n-1}}{A + Bx_n + x_{n-1}}$$

increases in x_{n-1} and decreases in x_n. By Theorem 2.5 it follows that the subsequences of the even and odd terms are eventually monotonic and because the solution is bounded these subsequences converge to finite limits. The proof is complete. □

Open Problem 4.1. *Assume that* (4.5) *holds*.

(i) *Determine the set of initial conditions* x_{-1}, x_0 *for which every solution of equation* (4.1) *converges to the equilibrium* $\bar{x}$.

(ii) *Determine the set of initial conditions* x_{-1}, x_0 *for which every solution of equation* (4.1) *converges to* (4.7).

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Appendix A. Table of the global character of the 49 special cases of Equation (1.1)

A bold faced **B** indicates that every solution of the equation in this special case is bounded and a bold faced **U** indicates that the equation in this special case has unbounded solutions in some range of its parameters and for some initial conditions. We also use the following designations:

ESB stands for '**every solution of the equation is bounded**'.

∃US stands for '**there exist unbounded solutions**'.

ESC$\bar{x}$ stands for '**every solution of the equation converges to the equilibrium point of the equation**'.

ESC stands for '**every solution of the equation converges to a finite limit**'.

EBSC$\bar{x}$ stands for '**every bounded solution of the equation converges to the equilibrium** $\bar{x}$'.

∃! **P_2-solution** stands for '**the equation has a unique prime period-two cycle**'.

ESP$_k$ stands for '**every solution of the equation is periodic with** (**not necessarily prime**) **period** k'.

ESCP$_k$ stands for '**every solution of the equation converges to a** (**not necessarily prime**) **period**-k solution'.

'**Has P**$_k$**-Tricho**' stands for '**the equation has a period**-k **trichotomy**'.

#1	$x_{n+1} = \alpha$	**B**	This equation is trivial
#2	$x_{n+1} = \alpha/x_n$	**B**	**ESP**$_2$
#3	$x_{n+1} = \alpha/(Cx_{n-1})$	**B**	**ESP**$_4$
#5	$x_{n+1} = (\beta/A)x_n$	**U**	This is a linear equation
#6	$x_{n+1} = \beta$	**B**	This equation is trivial
#7	$x_{n+1} = \frac{x_n}{x_{n-1}} >$	**B**	**ESP**$_6$
#9	$x_{n+1} = \gamma x_{n-1}$	**U**	This is a linear equation. This is the **only linear** equation with a $\mathbf{P}_2$ **− Tricho**
#10	$x_{n+1} = (x_{n-1}/x_n)$	**U**	Reducible to linear This equation is part of a $\mathbf{P}_2$ **− Tricho**; Theorem 5.1 in Part 1
#11	$x_{n+1} = \gamma$	**B**	This equation is trivial
#17	$x_{n+1} = \alpha/(A + Bx_n)$	**B**	This is a **Riccati** equation; **ESC**$\bar{x}$
#18	$x_{n+1} = \alpha/(A + Cx_{n-1})$	**B**	This is a **Riccati-type** equation; **ESC**$\bar{x}$
#20	$x_{n+1} = \alpha/(Bx_n + x_{n-1})$	**B**	**ESC**$\bar{x}$; ([36], p. 55) and [49]
#23	$x_{n+1} = \beta x_n/(A + Bx_n)$	**B**	This is a **Riccati** equation also known as the **Beverton–Holt** equation; **ESC**
#24	$x_{n+1} = \beta x_n/(A + Cx_{n-1})$	**B**	**Pielou's Equation**; **ESC**; [14,32,36,40,42]
#26	$x_{n+1} = \beta x_n/(Bx_n + Cx_{n-1})$	**B**	**ESC**$\bar{x}$; ([36], p. 58)
#29	$x_{n+1} = (x_{n-1})/(A + Bx_n)$	**U**	**Has** $\mathbf{P}_2$ **− Tricho**; [21,26,28–31,36,53,54,55]; Theorem 5.1 in Part 1
#30	$x_{n+1} = (x_{n-1})/(A + x_{n-1})$	**B**	This is a **Riccati-type** equation; ∃! $\mathbf{P_2}$**-solution and it is not LAS**; **ESCP**$_2$
#32	$x_{n+1} = (x_{n-1})/(Bx_n + x_{n-1})$	**B**	**ESCP**$_2$**-solution**; ([36], p. 60); ∃! $\mathbf{P_2}$**-solution when** $B \neq 1$ **which is LAS when** $B > 1$ **and infinitely many when** $B = 1$
#41	$x_{n+1} = \alpha + \beta x_n$	**U**	This is a linear equation
#42	$x_{n+1} = (\alpha + x_n)/x_n$	**B**	This is a **Riccati** equation; **ESC**$\bar{x}$
#43	$x_{n+1} = (\alpha + x_n)/(x_{n-1})$	**B**	**Lyness's Equation**; **No nontrivial solution has a limit**. [4–6,13,24,25,32,33,35], ([36], p. 70), [44–47,50,51,56]
#45	$x_{n+1} = \alpha + \gamma x_{n-1}$	**U**	This equation is linear
#46	$x_{n+1} = (\alpha + \gamma x_{n-1})/x_n$	**U**	This equation is part of a $\mathbf{P}_2$ **− Tricho**. ([36], p. 72); **EBSC**$\bar{x}$; Theorem 5.2 in Part 1
#47	$x_{n+1} = (\alpha + x_{n-1})/(x_{n-1})$	**B**	This is a **Riccati-type** equation; **ESC**$\bar{x}$
#53	$x_{n+1} = \beta x_n + \gamma x_{n-1}$	**U**	This is a linear equation
#54	$x_{n+1} = \beta + (x_{n-1}/x_n)$	**U**	**Has** $\mathbf{P}_2$ **− Tricho**; **The very first period-two trichotomy** [3] and ([36], p. 70); Theorem 5.1 in Part 1
#55	$x_{n+1} = (\beta x_n + x_{n-1})/(x_{n-1})$	**B**	**ESC**$\bar{x}$ ([36], p. 70)
#65	$x_{n+1} = (\alpha + \beta x_n)/(A + Bx_n)$	**B**	This is the **Riccati Equation** with **Riccati number** $\Re = (\beta A - \alpha B)/((\beta + A)^2) \leq (1/4)$ **ESC**$\bar{x}$; [9,23], ([36], p. 17)
#66	$x_{n+1} = (\alpha + x_n)/(A + x_{n-1})$	**B**	[32,33] and [36]; **Conjecture**: **ESC**$\bar{x}$
#68	$x_{n+1} = (\alpha + x_n)/(x_n + Cx_{n-1})$	**B**	([36], p. 82); **Conjecture**: **ESC**$\bar{x}$
#71	$x_{n+1} = (\alpha + \gamma x_{n-1})/(A + x_n)$	**U**	Has $\mathbf{P}_2$ **− Tricho**; ([36, p. 89) and [20]; Theorem 5.1 in Part 1
#72	$x_{n+1} = (\alpha + x_{n-1})/(A + x_{n-1})$	**B**	This is a **Riccati-type equation**; **ESC**$\bar{x}$
#74	$x_{n+1} = (\alpha + x_{n-1})/(Bx_n + x_{n-1})$	**B**	**ESCP**$_2$; **∃! P**$_2$**-solution when** $B > 1 + 4\alpha$ **and it is LAS.** [34] and ([36], p. 92)
#83	$x_{n+1} = (\beta x_n + \gamma x_{n-1})/(A + x_n)$	**U**	Has $\mathbf{P}_2$ **− Tricho**; ([36], p. 101) and [37]; Theorem 5.1 in Part 1
#84	$x_{n+1} = (\beta x_n + x_{n-1})/(A + x_{n-1})$	**B**	**ESC**; ([36], p. 109) and [38]
#86	$x_{n+1} = (\beta x_n + x_{n-1})/(Bx_n + x_{n-1})$	**B**	**ESCP**$_2$; **∃!** P$_2$**-solution and it is LAS**. ([36], p. 113), [39,41] and [48]
#101	$x_{n+1} = 1/(1 + Bx_n + Cx_{n-1})$	**B**	**ESC**$\bar{x}$; ([36], p. 71) and [49]
#105	$x_{n+1} = \beta x_n/(A + x_n + Cx_{n-1})$	**B**	**ESC**
#109	$x_{n+1} = x_{n-1}/(A + Bx_n + x_{n-1})$	**B**	**ESCP**$_2$**-solution** by Theorem 2.7 in Part 1. [12] and ([36], p. 133)
#117	$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1}$	**U**	This is a linear equation
#118	$x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/x_n$	**U**	Has $\mathbf{P}_2$ **− Tricho**; ([36], p. 137); Theorem 5.1 in Part 1
#119	$x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/x_{n-1}$	**B**	([36], p. 137); **Conjecture**: **ESC**$\bar{x}$ Can be transformed to #66 with $\alpha > A$ which is still a conjecture in this range of parameters
#141	$x_{n+1} = (\alpha + x_n)/(A + Bx_n + x_{n-1})$	**B**	([36], p. 141); **Conjecture**: **ESC**$\bar{x}$
#145	$x_{n+1} = (\alpha + x_{n-1})/(A + Bx_n + x_{n-1})$	**B**	**∃! P**$_2$ **solution and it is LAS**; **ESCP**$_2$. ([36], p. 149)

#153	$x_{n+1} = (\beta x_n + x_{n-1})/(A + Bx_n + x_{n-1})$	**B**	**∃! P_2 solution and we conjecture that it is LAS**. **Conjecture**: **ESCP_2** ([36], p. 158)
#165	$x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + x_n)$	**U**	**Has P_2 − Tricho**; ([36], p. 167); Theorem 5.1 in Part 1. **EBSC**$\bar{x}$ when $\gamma > \beta + A$
#166	$x_{n+1} = (\alpha + x_n + \gamma x_{n-1})/(A + x_{n-1})$	**B**	**Conjecture**: **ESC**$\bar{x}$. ([36], p. 172)
#168	$x_{n+1} = (\alpha + x_n + \gamma x_{n-1})/(Bx_n + x_{n-1})$	**B**	**∃! P_2 solution and we conjecture that it is LAS**. **ESCP_2**; ([36], p. 17)
#201	$x_{n+1} = (\alpha + \beta x_n + x_{n-1})/(A + Bx_n + x_{n-1})$	**B**	**∃! P_2 solution if and only if** $\beta + A < 1$ **and** $4\alpha < (1 - \beta - A)[B(1 - \beta - A) - (1 - 3\beta - A)]$ **and we conjecture that is LAS**. **Conjecture**: $\bar{x}$ **is GAS when** either $\beta + A \geq 1$ or $4\alpha < (1 - \beta - A)[B(1 - \beta - A) - (1 - 3\beta - A)]$. **Conjecture**: **ESC$P_2$**